Lecture 21:

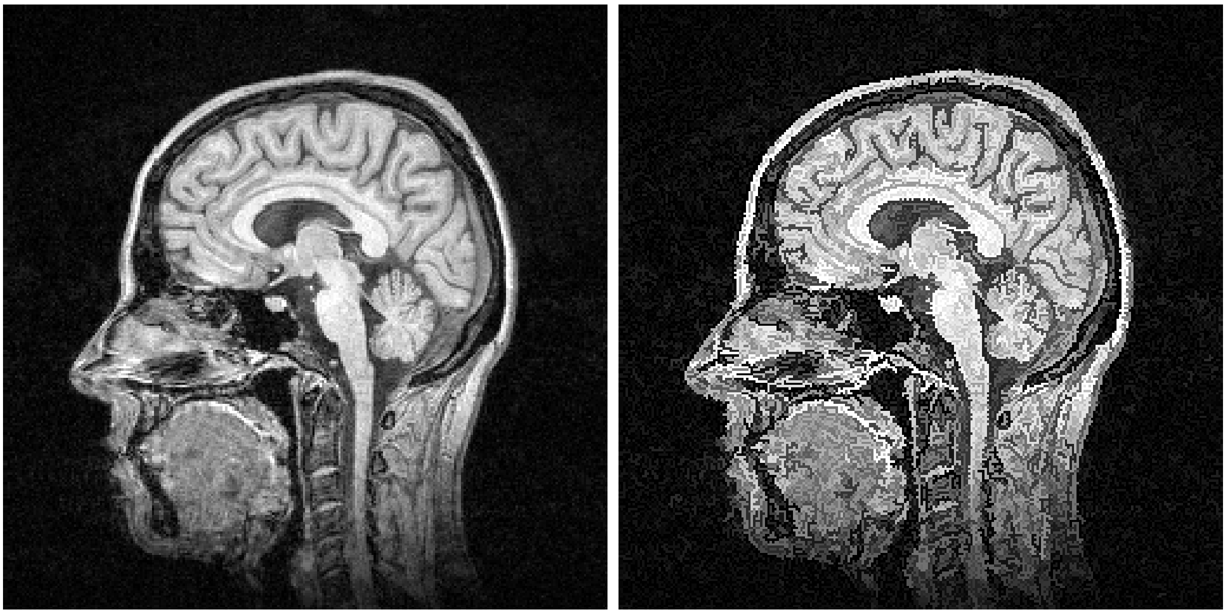
Continuous-Scale Morphology II: Shock Filters and Nonflat Morphology

Contents

Shock Filtering

- morphological deblurring technique
- first discrete formulation by Kramer/Bruckner (1975), first PDE-based decription by Osher/Rudin (1990)
- dilation around maxima (regions where $\Delta u < 0$), erosion around minima (regions where $\Delta u > 0$)
- simplest representative:
$$\partial_t u \;=\; -\mathrm{sgn}(\Delta u)\, |\nabla u|$$
- creates shocks at edge locations where $\Delta u = 0$.
- no well-posedness theory known
- satisfies a discrete extremum principle
- Experiments show that segmentation-like nontrivial steady states exist.
- main disadvantage: noise sensitive

Shock Filtering (2)



(a) **Left:** Original image, $\Omega = (0, 256)^2$. (b) **Right:** Result when applying a shock filter, $t = 100$.

Shock Filtering (3)

- Osher/Rudin 1990:
  - suggested that $u_{\eta\eta}$ with $\eta \parallel \nabla u$ can be a better edge detector than Δu:

$$\partial_t u = -\operatorname{sgn}(u_{\eta\eta})\, |\nabla u|$$

- Alvarez/Mazorra 1994:
  - replaced $u_{\eta\eta}$ by $v_{\eta\eta}$ with a Gaussian-smoothed image $v := K_\sigma * u$:

$$\partial_t u = -\operatorname{sgn}(v_{\eta\eta})\, |\nabla u|$$

  - more robust under noise
- Weickert 2003:
  - replaced $v_{\eta\eta}$ by $v_{\mathbf{ww}}$ where $\mathbf{w}$ is the normalised dominant eigenvector of the structure tensor $J_\rho(\nabla u)$:

$$\partial_t u = -\operatorname{sgn}(v_{\mathbf{ww}})\, |\nabla u|$$

  - enhances line-like structures *(coherence-enhancing shock filtering)*.

Shock Filtering (4)

(a) **Left:** Fingerprint image, 186×186 pixels. (b) **Middle:** Steady state using the Alvarez–Mazorra shock filter with $\sigma = 1.5$. (c) **Right:** Steady state using coherence-enhancing shock filtering with $\sigma = 1.5$ and $\rho = 5$.

Shock Filtering (5)

Increasing the structure scale σ in coherence-enhancing shock filtering may create artistic effects. (a) **Left:** Mandrill, 512×512 pixels. (b) **Middle:** Coherence-enhancing shock filtering, $\sigma = 2$, $\rho = 5$, $t = 10$. (c) **Right:** Ditto with $\sigma = 4$.

Shock Filtering (6)

(a) **Left:** Original image of Professor Albrecht, 480×640 pixels. (b) **Right:** Coherence-enhancing shock filtering, $\sigma = 2$, $\rho = 4$, $t = 50$.

Nonflat Morphology (1)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Nonflat Morphology

- **Motivation:**
  - Morphological dilation/erosion with a structuring element resembles linear convolution with a box function.
  - However, convolution also allows nonflat functions as kernels.
  - Is there a morphological equivalent, such as a nonflat structuring element ?
- Consider the *structuring function (SF)*

$$b(\mathbf{x}) : \quad \mathbb{R}^2 \to \mathbb{R}$$

  instead of a structuring element B.

- Morphological operations on some image $f(\mathbf{x})$ with a structuring function $b(\mathbf{x})$:

$$\textit{dilation:} \qquad (f \oplus b)\,(\mathbf{x}) \;:=\; \sup\,\{f(\mathbf{x}-\mathbf{y}) + b(\mathbf{y}) \mid \mathbf{y}\in\mathbb{R}^2\}.$$

$$\textit{erosion:} \qquad (f \ominus b)\,(\mathbf{x}) \;:=\; \inf\,\{f(\mathbf{x}+\mathbf{y}) - b(\mathbf{y}) \mid \mathbf{y}\in\mathbb{R}^2\}.$$

- The use of structuring functions renounces morphological invariance.

Connections to Flat Morphology and Structuring Elements

- Identifying a structuring element B with the "flat" structuring function

$$b(\mathbf{x}) \;:=\; \begin{cases} 0 & (\mathbf{x}\in B), \\ -\infty & (\mathbf{x}\notin B). \end{cases}$$

  transfers the novel definition

$$\text{dilation:} \qquad (f \oplus b)\,(\mathbf{x}) \;:=\; \sup\,\{f(\mathbf{x}-\mathbf{y}) + b(\mathbf{y}) \mid \mathbf{y}\in\mathbb{R}^2\}.$$

$$\text{erosion:} \qquad (f \ominus b)\,(\mathbf{x}) \;:=\; \inf\,\{f(\mathbf{x}+\mathbf{y}) - b(\mathbf{y}) \mid \mathbf{y}\in\mathbb{R}^2\}.$$

  to the original definition:

$$\text{dilation:} \qquad (f \oplus b)\,(\mathbf{x}) \;:=\; \sup\,\{f(\mathbf{x}-\mathbf{y}) \mid \mathbf{y}\in B\},$$

$$\text{erosion:} \qquad (f \ominus b)\,(\mathbf{x}) \;:=\; \inf\,\{f(\mathbf{x}+\mathbf{y}) \mid \mathbf{y}\in B\}.$$

This generalisation does not harm, but is is useful?

M I A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

The Slope Transform

- The new definition of dilation is commutative:

$$f \oplus b \;=\; b \oplus f$$

  Similar to convolution: $f * b \;=\; b * f$

- Dilation in morphology resembles convolution in linear system theory (LST).
  In LST, the Fourier transform maps convolution to multiplication.
  Is there a similar transformation for dilation in morphology?

- Yes, if one considers *tangential dilation*:

$$(f \,\check{\oplus}\, b)(\mathbf{x}) \;:=\; \operatorname*{stat}_{\mathbf{y}} \Big(f(\mathbf{x}-\mathbf{y}) + b(\mathbf{y}) \Big)$$

  where the *stationary values* $\operatorname*{stat}_{\mathbf{z}} f(\mathbf{z}) := \{f(\mathbf{z}) \mid \nabla f(\mathbf{z}) = \mathbf{0}\}$ contain the grey values at extrema.

- Tangential dilation is equivalent to usual dilation if f and b are concave (since the extrema are maxima then).

Nonflat Morphology (4)

- Dorst/van den Boomgaard and Maragos (1994) showed:
  The morphological equivalent to the Fourier transform is the *slope transform*

  $$\mathcal{S}[f](\boldsymbol{\omega}) := \operatorname*{stat}_{\mathbf{x}} \left(f(\mathbf{x}) - \boldsymbol{\omega}^\top \mathbf{x} \right)$$

  Note the almost logarithmic connection to the Fourier transform (FT):

  $$\mathcal{F}[f](\boldsymbol{\omega}) := \int \left(f(\mathbf{x}) \, \exp(-i\boldsymbol{\omega}^\top \mathbf{x}) \right) d\mathbf{x}$$

- The slope transform maps tangential dilation to addition:

  $$\mathcal{S}[f \,\check{\oplus}\, b] \;=\; \mathcal{S}[f] + \mathcal{S}[b]$$

  while the Fourier transform maps convolution to multiplication:

  $$\mathcal{F}[f * b] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[b]$$

Nonflat Morphology (5)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

- The backtransformation to the slope transformation is given by

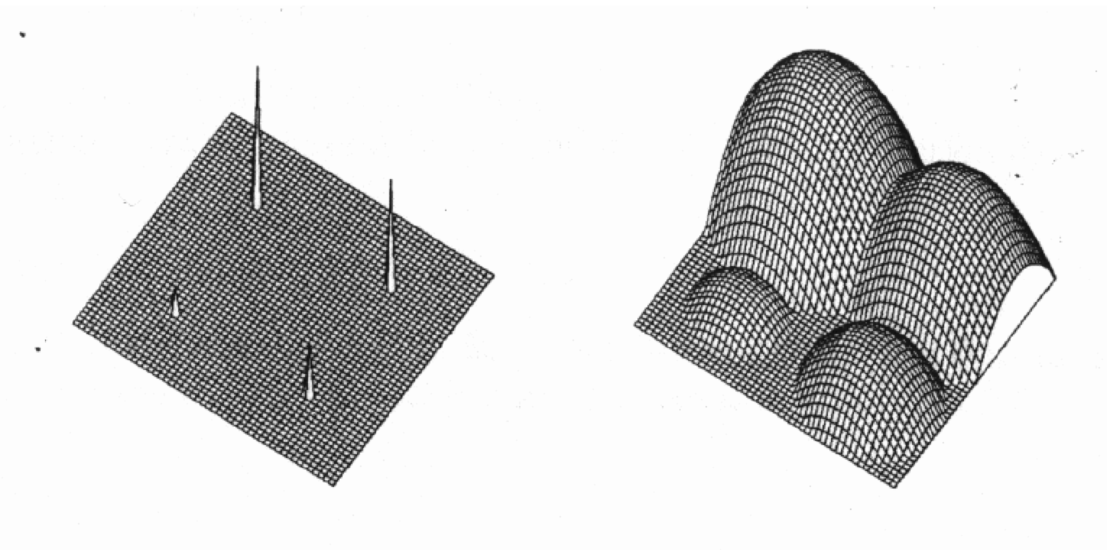
  $$f(\mathbf{x}) \;=\; \operatorname*{stat}_{\boldsymbol{\omega}} \left(\mathcal{S}[f](\boldsymbol{\omega}) + \boldsymbol{\omega}^\top \mathbf{x} \right).$$

  Backtransformation to the Fourier transformation:

  $$f(\mathbf{x}) \;=\; \frac{1}{2\pi} \int \left(\mathcal{F}[f](\boldsymbol{\omega}) \, \exp(i\boldsymbol{\omega}^\top \mathbf{x}) \right) d\boldsymbol{\omega}.$$

Is There a Morphological Equivalent to Gaussian Convolution?

- In LST, convolution with Gaussians plays a fundamental role:
  - Gaussians remain Gaussians under the Fourier transform.
  - Gaussians are the only separable and rotationally invariant convolution kernels.
- Similar results in morphology (van den Boomgaard 1992):
  - Parabolas remain parabolas under the slope transform.
  - Parabolas are the only separable and rotationally invariant structuring functions.



A 2-D signal and its dilation with a paraboloid as structuring function. **Author:** van den Boomgaard (1992).

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Nonflat Morphology (8)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

PDE Formulation

- van den Boomgaard (1992) showed:
  Dilation of $f(\mathbf{x})$ with a *quadratic structuring function (QSF)*

$$b(\mathbf{x}, t) \;=\; -\frac{|\mathbf{x}|^2}{4t} \qquad (t > 0)$$

  is equivalent to solving the differential equation

$$\partial_t u = |\nabla u|^2$$

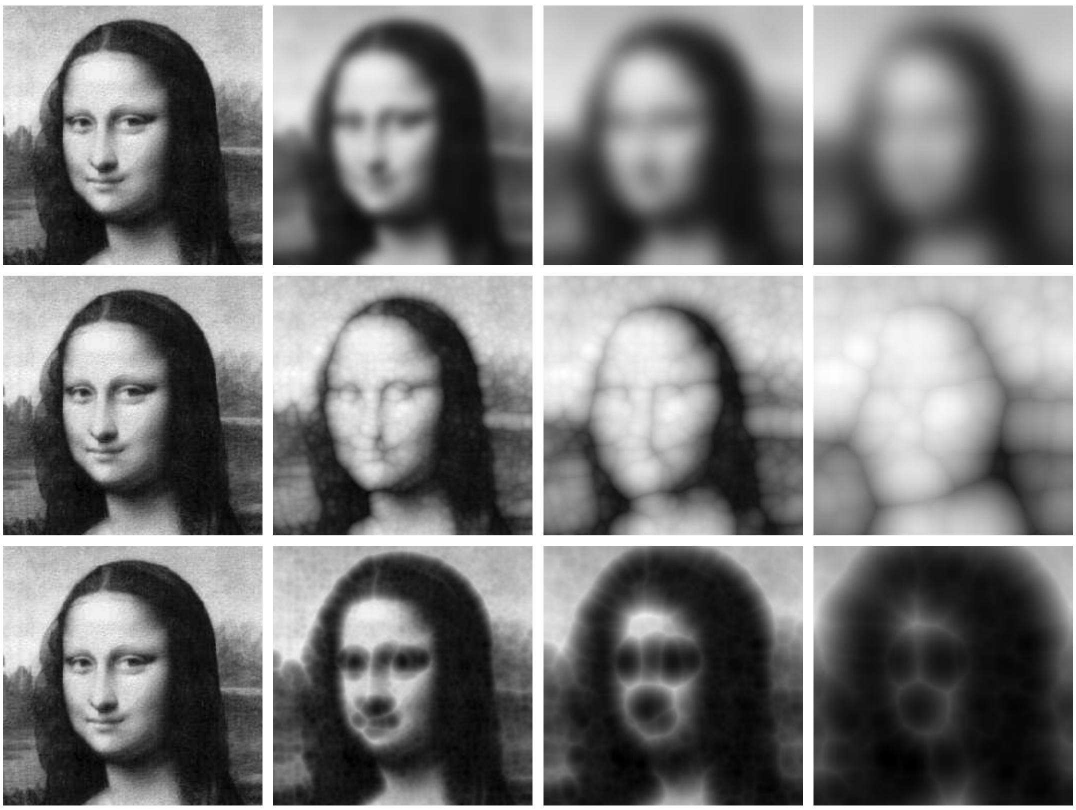
  with initial value $f(\mathbf{x})$.

- Erosion with a QSF is governed by

$$\partial_t u = -|\nabla u|^2.$$

Nonflat Morphology (9)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25



(a) **Top row:** Evolution under linear diffusion. From left to right: $t = 0,\ 5,\ 10,\ 15$. (b) **Middle row:** Dilation with a QSF: $t = 0,\ 0.25,\ 1,\ 4$. (c) **Bottom row:** Erosion with a QSF: $t = 0,\ 0.25,\ 1,\ 4$.

Nonflat Morphology (10)

A Simple Algorithm for Dilations with QSFs

- separability: only 1D dilations necessary
- does not require to solve a PDE in an iterative manner
- uses direct implementation of the definition of dilation via the sup-operator
- exact, noniterative method for arbitrarily large times t

```
for (i=0; i<=N-1; i++)
   {
   max = f[i];
   for (j=0; j<=N-1; j++)
      {
      help = f[j] - (i - j) * (i - j) / (4.0 * t);
      if (help > max) max = help;
      }
   u[i] = max;
   }
```

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Nonflat Morphology (11)

Application of QSFs: Euclidean Distance Transformation

(van den Boomgaard 1992)

- given: binary image $f(\mathbf{x})$
- set foreground to 0 and background to ∞
- erode with a quadratic structuring function $b(\mathbf{x}, t) = -\mathbf{x}^2$
- gives the squared distance function
- noniterative, “exact” method
- tip for better visualisation:
  periodic repetition of the grey scale after a few pixels

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Nonflat Morphology (12)

(a) **Top left:** Image with a single object pixel in the centre. (b) **Top right:** Euclidean distance transformation of (a). The greyscale of the distances is repeated periodically after 16 pixels. (c) **Bottom left:** Test image. (d) **Bottom right:** Euclidean distance transformation of (c).

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Relations to Linear System Theory (1)

Relations to Linear System Theory

(Burgeth/W. 2005)

There is an almost logarithmic connection between

linear system theory	and	*morphological system theory*
$\mathcal{F}[f](u)$	$\longleftrightarrow$	$\mathcal{S}[f](u)$
$\mathcal{F}[f * b] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[b]$	$\longleftrightarrow$	$\mathcal{S}[f \,\check{\oplus}\, b] \;=\; \mathcal{S}[f] + \mathcal{S}[b]$
$\frac{1}{(2\pi\sigma^2)^{n/2}}\, e^{-\frac{\mathbf{x}^\top \mathbf{x}}{2\sigma^2}}$	$\longleftrightarrow$	$-\frac{\mathbf{x}^\top \mathbf{x}}{4t}$

What is the reason for this connection?
Is there a common basis for the two system theories?

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

The (max,+)- and the (min,+)-Algebra

Define $(\max,+)$-*algebra* $\mathbb{R}_{max}$ and $(\min,+)$-*algebra* $\mathbb{R}_{min}$ via the table

name	set	addition	multiplication
standard algebra $\mathbb{R}$	$\mathbb{R}$	$+$	$\times$
$(\max,+)$-algebra $\mathbb{R}_{max}$	$\mathbb{R} \cup \{-\infty\}$	max	$+$
$(\min,+)$-algebra $\mathbb{R}_{min}$	$\mathbb{R} \cup \{+\infty\}$	min	$+$

M I A

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25

Regarding the signal f as a mapping into these algebras replaces the conventional definition of convolution

$$(f * g)(\mathbf{x}) \;=\; \int_{\mathbb{R}^n} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{x}$$

by *new definitions of morphological convolutions*:

$$(f *_d g)(\mathbf{x}) \;:=\; \sup_{\mathbf{y}\in\mathbb{R}^n} \big(f(\mathbf{x}-\mathbf{y}) + g(\mathbf{y}) \big),$$
$$(f *_e g)(\mathbf{x}) \;:=\; \inf_{\mathbf{y}\in\mathbb{R}^n} \big(f(\mathbf{x}-\mathbf{y}) + g(\mathbf{y}) \big).$$

They are closely connected to dilation $\oplus$ and erosion $\ominus$ (with $\overline{g}(\mathbf{x}) := -g(-\mathbf{x})$):

$$(f \oplus g)(\mathbf{x}) \;=\; \sup_{\mathbf{y}\in\mathbb{R}^n} (f(\mathbf{x}-\mathbf{y}) + g(\mathbf{y})) \;=\; (f *_d g)(\mathbf{x}),$$
$$(f \ominus g)(\mathbf{x}) \;=\; \inf_{\mathbf{y}\in\mathbb{R}^n} (f(\mathbf{x}+\mathbf{y}) - g(\mathbf{y})) \;=\; (f *_e \overline{g})(\mathbf{x}).$$

Thus, morphology is linear system theory in another algebra.

Summary

- Shock filters are morphological image enhancement methods that use dilation around maxima and erosion around minima.
- They may be modified in many ways, allowing even to enhance coherent, flow-like structures.
- The classical "flat" morphology with structuring elements may be generalised to nonflat morphology with structuring functions.
- Morphology with quadratic structuring functions provides an elegant way to compute Euclidean distance transformations.
- The resulting morphological system theory resembles linear system theory:
  - The morphological equivalent to convolution is (tangential) dilation.
  - The morphological equivalent to the Fourier transform is the slope transform.
  - The morphological equivalent to Gaussian convolution kernels are quadratic structuring functions.
- Morphology may be regarded as linear system theory in another algebra: the maxplus algebra resp. minplus algebra.

References

- H. P. Kramer and J. B. Bruckner: Iterations of a non-linear transformation for enhancement of digital images. *Pattern Recognition*, Vol. 7, 53–58, 1975.
  (introduced shock filters in a discrete way, without any PDE formulation).
- S. Osher, L. Rudin: Feature-oriented image enhancement using shock filters. *SIAM Journal on Numerical Analysis*, Vol. 27, 919–949, 1990.
  (first PDE formulation for shock filters)
- L. Alvarez and L. Mazorra: Signal and image restoration using shock filters and anisotropic diffusion. *SIAM Journal on Numerical Mathematics*, Vol. 31, 590–605, 1994.
  (stabilisation attempts for shock filters)
- J. Weickert: Coherence-enhancing shock filters. In B. Michaelis, G. Krell (Eds.): *Pattern Recognition*. Lecture Notes in Computer Science, Vol. 2781, Springer, Berlin, 1–8, 2003. (`http://www.mia.uni-saarland.de/publications.shtml`).
  (introduced coherence-enhancing shock filters)
- R. van den Boomgaard: The morphological equivalent of the Gauss convolution. *Nieuw Archief Voor Wiskunde*, Vol. 10, No. 3, 219–236, 1992.
  (nonflat morphology with parabolic structuring functions)
- L. Dorst R. van den Boomgaard: Morphological signal processing and the slope transform. *Signal Processing*, Vol. 38, No. 1, 79–98, 1994.
  (introduced the slope transform together with Maragos)

References (2)

- P. Maragos: Morphological systems: Slope transforms and max-min difference and differential equations. *Signal Processing*, Vol. 38, No. 1, 57–77, 1994.
  (introduced the slope transform together with Dorst / and den Boomgaard)
- B. Burgeth, J. Weickert: An explanation for the logarithmic connection between linear and morphological system theory. *International Journal of Computer Vision*, Vol. 64, No. 2/3, 157–169, Sept. 2005.
  `(http://www.mia.uni-saarland.de/publications.shtml)`
  (maxplus algebra and Cramer transform to unify linear and morphological systems)