Lecture 20:

Continuous-Scale Morphology I: Basic Ideas

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Classical Morphology

- image analysis based on shapes
- foundations by Matheron and Serra in the sixties at ENS des Mines de Paris in Fontainebleau
- processes images by computing maxima or minima within a neighbourhood specified by a so-called structuring element
- many applications: cell biology, quality control, remote sensing, mineralogy, ...
- typically described in terms of algebraic set theory, but also PDE formulations (Brockett/Maragos 1992, van den Boomgaard 1992, Arehart/Vincent/Kimia 1993, Alvarez/Guichard/Lions/Morel 1993)

Classical Morphology (2)

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Morphology with a Structuring Element B:

- typical structuring elements B: discs, squares, or ellipses.
- two basic morphological processes:

$$\begin{aligned} \textit{dilation:} \quad & (f \oplus B)(\mathbf{x}) := \sup \{ f(\mathbf{x}-\mathbf{y}) \mid \mathbf{y} \in B \}, \\ \textit{erosion:} \quad & (f \ominus B)(\mathbf{x}) := \inf \{ f(\mathbf{x}+\mathbf{y}) \mid \mathbf{y} \in B \}. \end{aligned}$$

- allow to assemble other morphological operations, e.g.

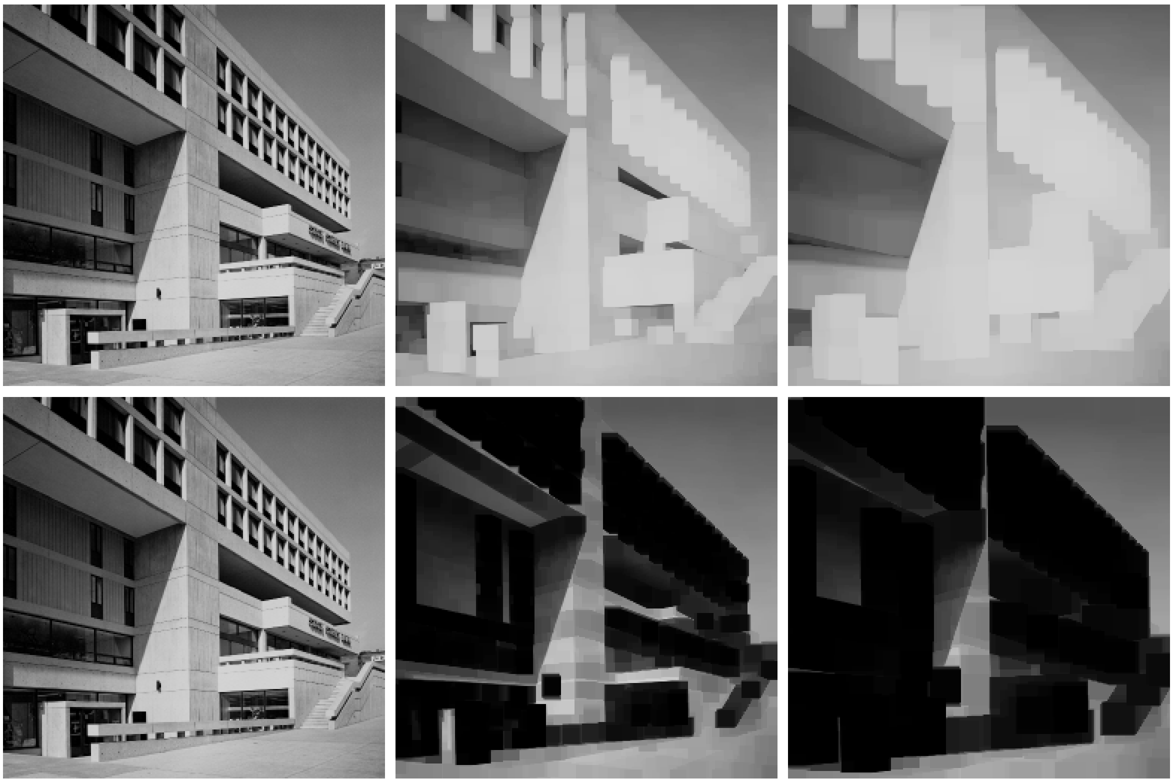
$$\begin{aligned} \textit{opening:} \quad & (f \circ B)(\mathbf{x}) := ((f \ominus B) \oplus B)(\mathbf{x}), \\ \textit{closing:} \quad & (f \bullet B)(\mathbf{x}) := ((f \oplus B) \ominus B)(\mathbf{x}). \end{aligned}$$

Morphological Invariance:

- invariant under monotoneously increasing grey level rescalings:
  infimum or supremum is attained in the same pixel
- This means that contrast does not matter !

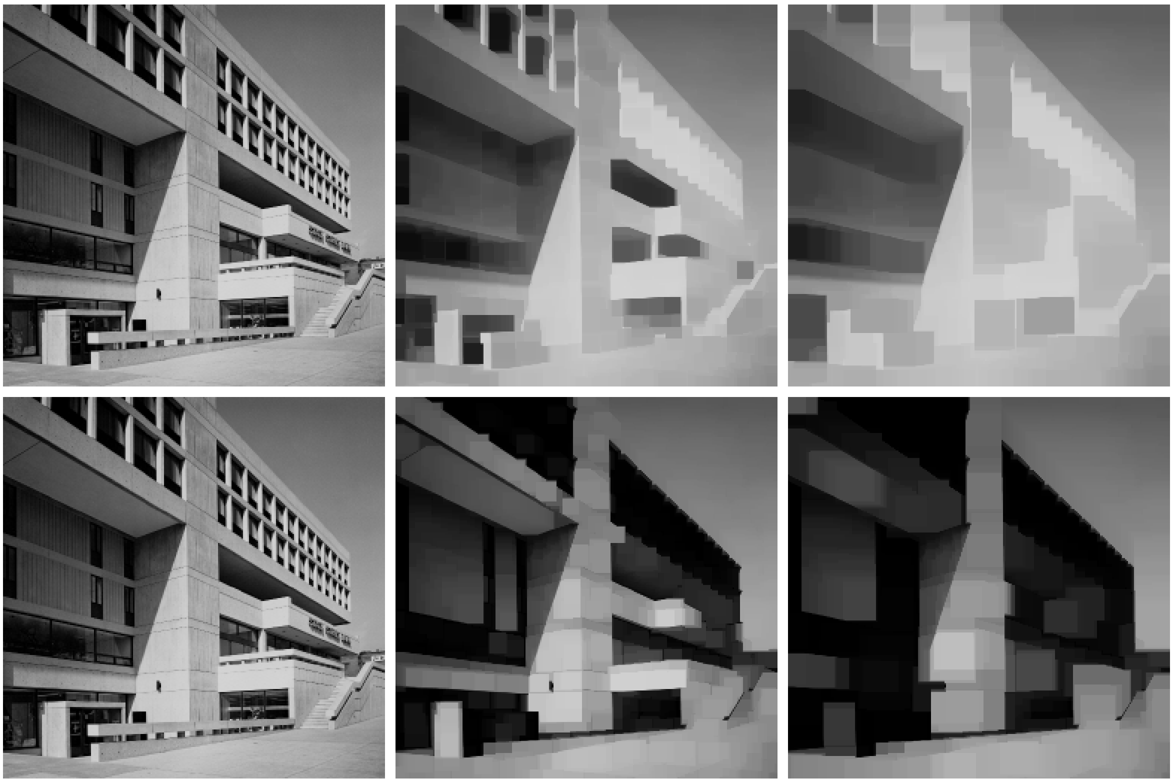
Classical Morphology (3)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23



(a) **Top row, from left to right:** Dilation of a greyscale image (256×256 pixels) with a square of length 11 and 21 pixels. (b) **Bottom row, from left to right:** Corresponding erosion.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23



(a) **Top row, from left to right:** Closing of a greyscale image (256×256 pixels) with a square of length 11 and 21 pixels. (b) **Bottom row, from left to right:** Corresponding opening.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23

Morphological Curve Evolution

- Consider binary image object with boundary curve $\mathbf{c_0}$
  (with some arbitrary parametrisation $\mathbf{c_0} = (x_1(p), x_2(p))^\top$).
  Dilate it with disc-shaped structuring elements of continuously increasing radius t.

- Then the resulting curve $\mathbf{c}(t)$ follows the evolution

$$\partial_t \mathbf{c} = \mathbf{n}$$

  with initial condition

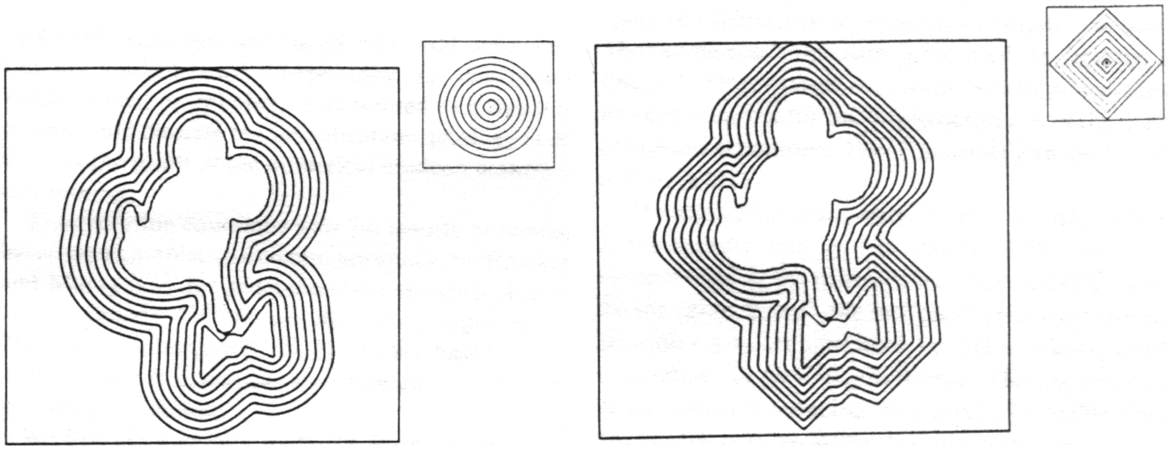
$$\mathbf{c}(0) = \mathbf{c}_0$$

  and stopping time t.
  It describes propagation in outer normal direction $\mathbf{n}$ with speed 1.

- Erosion is propagation in inner normal direction $-\mathbf{n}$:

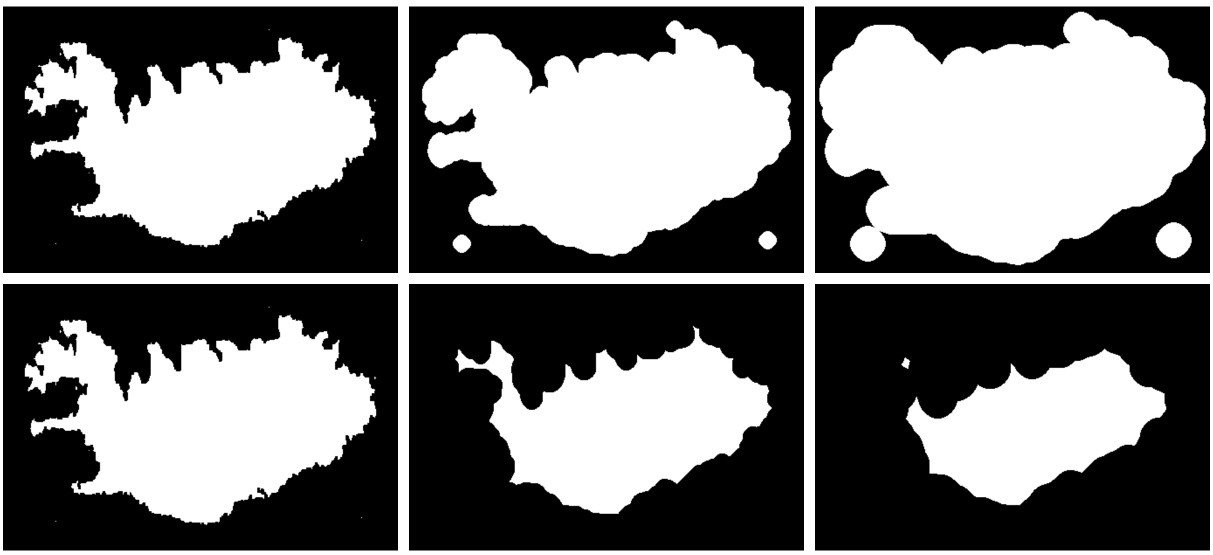
$$\partial_t \mathbf{c} = -\mathbf{n}$$

Morphological Curve Evolution (2)



Contour evolution of the shape "Mickey" using dilations with increasing structuring element. (a) **Left:** Disc as structuring element. (b) **Right:** Diamond as structuring element. **Author:** Sapiro et al. 1993.

Morphological Curve Evolution (3)



(a) **Top row, from left to right:** Dilation of the countour of Iceland with a disc-shaped structuring element of radius 0, 10 and 20. Image size: 452×305 pixels. (b) **Bottom row, from left to right:** Erosion instead of dilation.

From Curve Evolution to Image Evolution

- Consider some curve evolution

$$\partial_t \mathbf{c} \;=\; \beta(\kappa) \cdot \mathbf{n}.$$

where the speed function β may even depend on the curvature κ.

- Apply *level set* ideas:
Embed $\mathbf{c}$ as a level line into some sufficiently smooth scalar-valued image u: consider e.g. the *signed distance function*

$$u(\mathbf{x}, t) := \begin{cases} \operatorname{dist}(\mathbf{x}, \mathbf{c}(t)) & (\mathbf{x} \text{ inside } \mathbf{c}) \\ -\operatorname{dist}(\mathbf{x}, \mathbf{c}(t)) & (\text{else}). \end{cases}$$

- The zero-crossings of u are given by $\mathbf{c}(t) = \binom{x_1(t)}{x_2(t)}$:

$$u(\mathbf{c}, t) = 0.$$

- Differentiate

$$0 \;=\; u(\mathbf{c}, t) \;=\; u(x_1(t), x_2(t), t)$$

with respect to t:

$$\begin{aligned} 0 &= \frac{du}{dt} \\ &= \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t} + \frac{\partial u}{\partial t} \\ &= (\nabla u)^\top \partial_t \mathbf{c} + \partial_t u. \end{aligned}$$

- With the outer normal vector $\mathbf{n} = -\frac{\nabla u}{|\nabla u|}$, one obtains the image evolution

$$\partial_t u \;=\; -(\nabla u)^\top \partial_t \mathbf{c} \;=\; -(\nabla u)^\top \beta(\kappa)\, \mathbf{n} \;=\; \beta(\kappa)\, |\nabla u|.$$

Any curve evolution $\partial_t \mathbf{c} = \beta\, \mathbf{n}$ can be embedded as a level set into the image evolution $\partial_t u = \beta\, |\nabla u|$. This is the basis of the so-called level set methods. Conversely, one can also show that any image evolution of type $\partial_t u = \beta |\nabla u|$ leads to the evolution equation $\partial_t \mathbf{c} = \beta\, \mathbf{n}$ for its level curves.

PDEs for Continuous-Scale Morphology

Image Evolution for Discs as Structuring Elements

- Curve evolutions

$$\partial_t \mathbf{c} \;=\; \pm \mathbf{n}$$

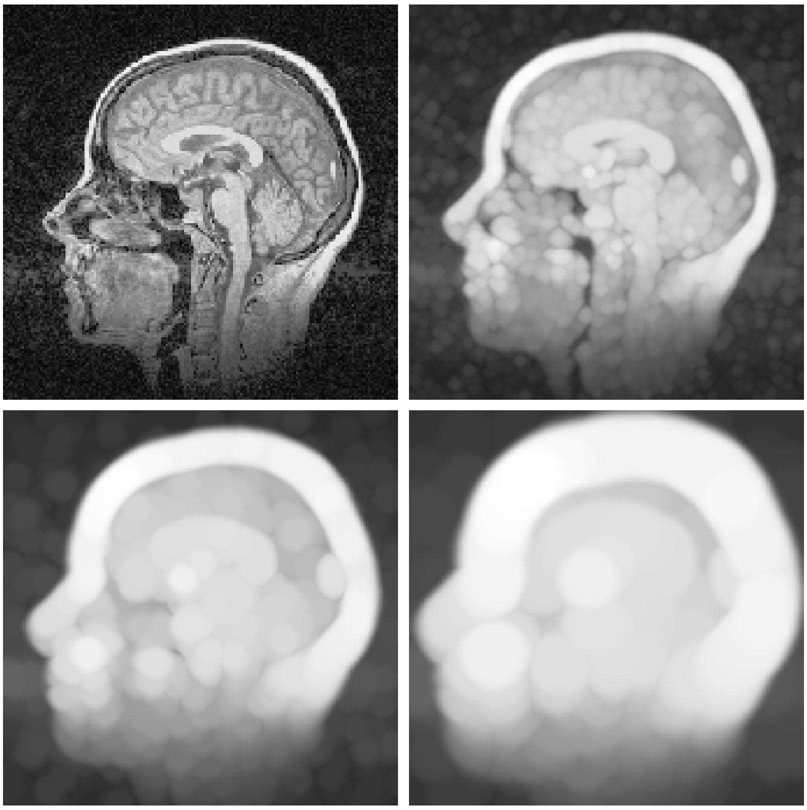
lead to image evolutions

$$\partial_t u \;=\; \pm |\nabla u|.$$

- plus sign for dilation, minus sign for erosion.
- Similar equations exist for other convex structuring elements (ellipses, squares, diamonds, ...).

Fundamental Difference to Diffusion Equations ("Parabolic" PDEs):

- only first order derivatives in space
- wave-like propagation behaviour ("hyperbolic" PDE)
- no specific time direction that is more dangerous than the other: forward and backward evolutions are equally stable.



Scale-space behaviour of dilation with a disc. (a) **Top left:** Original image, $\Omega = (0, 236)^2$. (b) **Top right:** $t = 4$. (c) **Bottom left:** $t = 10$. (d) **Bottom right:** $t = 20$.

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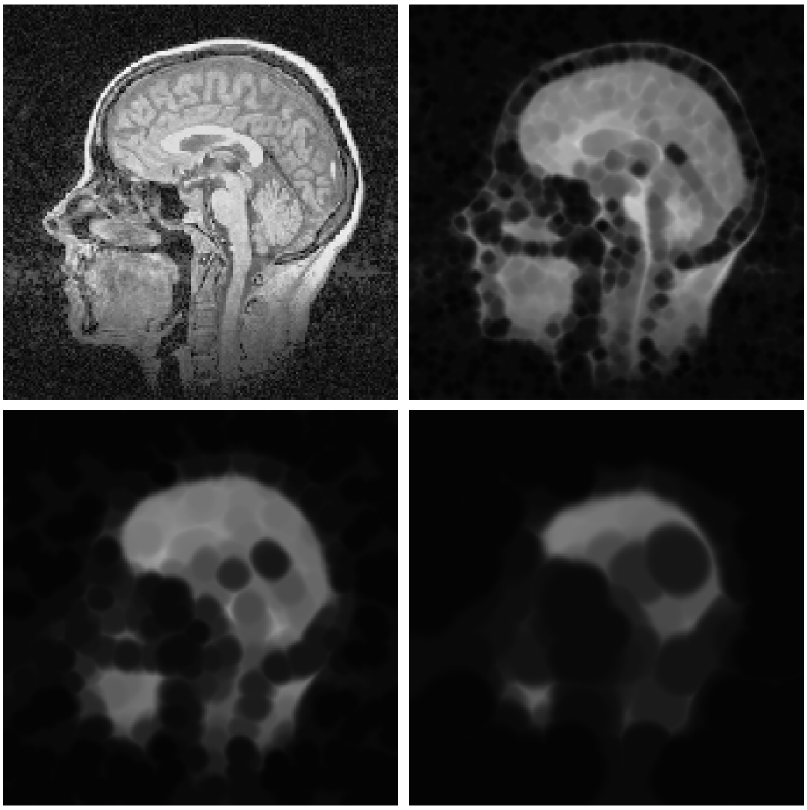

Scale-space behaviour of erosion with a disc. (a) **Top left:** Original image, $\Omega = (0, 236)^2$. (b) **Top right:** $t = 4$. (c) **Bottom left:** $t = 10$. (d) **Bottom right:** $t = 20$.

Theoretical Results for the Image Evolution

- no solution concept with strong regularity results
  (such as C^∞ solutions for diffusion filters)
- Alvarez/Guichard/Lions/Morel 1993:
  - well-posedness in terms of so-called viscosity solutions
  - extremum principle:

$$\inf f \;\le\; u(\mathbf{x}, t) \;\le\; \sup f \qquad \forall\, \mathbf{x},\ \forall\, t > 0.$$

  - L^∞-stability w.r.t. perturbations of the initial data:

$$\|u(.,t) - \tilde{u}(.,t)\|_{L^\infty(\mathbb{R}^2)} \;\le\; \|f - \tilde{f}\|_{L^\infty(\mathbb{R}^2)}.$$

- important scale-space property (Jackway 1992):
  – erosion/dilation reduces number of local minima/maxima;
  – preserves their location;
  – advantage over diffusion filtering for dimensions ≥ 2.
- disadvantage: noise sensitivity
  (dilation converges to global supremum, erosion to global infimum)

Numerical Methods (Osher/Sethian 1988)

- inspired by numerical schemes for hyperbolic conservation laws;
- *upwind schemes:* one-sided differences in direction from which information comes

Dilation

The dilation equation

$$\partial_t u \;=\; \sqrt{(\partial_{x_1} u)^2 + (\partial_{x_2} u)^2}$$

is approximated by

$$\begin{aligned}\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} \;=\;& \Bigg(\left(\min\left(\frac{u_{i,j}^n - u_{i-1,j}^n}{h_1}, 0 \right) \right)^2 + \left(\max\left(\frac{u_{i+1,j}^n - u_{i,j}^n}{h_1}, 0 \right) \right)^2 \\ &+ \left(\min\left(\frac{u_{i,j}^n - u_{i,j-1}^n}{h_2}, 0 \right) \right)^2 + \left(\max\left(\frac{u_{i,j+1}^n - u_{i,j}^n}{h_2}, 0 \right) \right)^2 \Bigg)^{1/2}\end{aligned}$$

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Erosion

The erosion equation

$$\partial_t u \;=\; -\sqrt{(\partial_{x_1} u)^2 + (\partial_{x_2} u)^2}$$

is approximated by

$$\begin{aligned}\frac{u_{i,j}^{n+1} - u_{i,j}^n}{\tau} \;=\;& -\Bigg(\left(\max\left(\frac{u_{i,j}^n - u_{i-1,j}^n}{h_1}, 0 \right) \right)^2 + \left(\min\left(\frac{u_{i+1,j}^n - u_{i,j}^n}{h_1}, 0 \right) \right)^2 \\ &+ \left(\max\left(\frac{u_{i,j}^n - u_{i,j-1}^n}{h_2}, 0 \right) \right)^2 + \left(\min\left(\frac{u_{i,j+1}^n - u_{i,j}^n}{h_2}, 0 \right) \right)^2 \Bigg)^{1/2}\end{aligned}$$

Stability in terms of an extremum principle for $\tau \le 1/2$.

Numerical Methods (3)

Advantages of PDE-Based Schemes Over Set-Theoretic Schemes

- good results for non-digitally scalable structuring elements:
  discs, ellipses
- sub-pixel accuracy

Disadvantages

- slower
- dissipative effects: shocks become blurred

Numerical Methods (4)

Remedy to Reduce Dissipative Effects

(Breuß/W. 2006)

- consider so called *flux-corrected transport (FCT) schemes*
- reduce diffusive effects of upwind schemes by a stabilised inverse diffusion step
- yield sharp edges without violating a maximum–minimum principle

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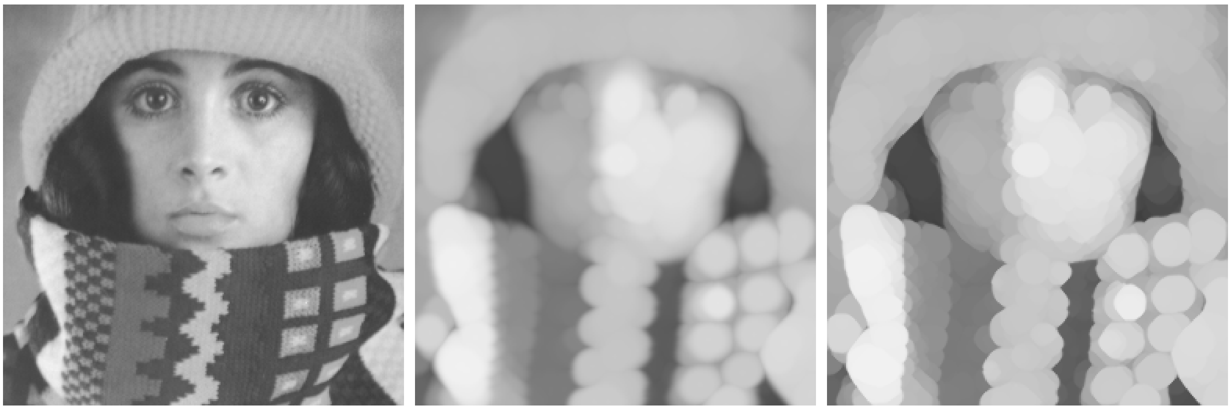

Comparison of numerical schemes for continuous-scale morphology. (a) **Left:** Original image, 256×256 pixels. (b) **Middle:** Dilation with a disc, using the Osher-Sethian upwind scheme ($t = 10$). (c) **Right:** Same experiment, but with the Breuß-Weickert FCT scheme.

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Summary

- PDE formulations of dilation / erosion exist for scalable convex structuring elements
- Curve evolutions can be translated into morphological image evolutions and vice versa.
- PDE-based morphological schemes are advantegeous for non-digitally scalable structuring elements (discs, ellipses) and sub-pixel accuracy.
- Continuous-scale morphology has attractive well-posedness and scale-space results, but is noise sensitive.
- The Osher–Sethian upwind schemes may suffer from blurring artifacts.
  FCT schemes perform better in this respect.

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References

- G. Sapiro, R. Kimmel, D. Shaked, B. B. Kimia, A. M. Bruckstein, Implementing continuous-scale morphology via curve evolution, *Pattern Recognition*, Vol. 26, 1363–1372, 1993.
  (well-readable paper on the foundations and numerical aspects)
- L. Alvarez, F. Guichard, P.-L. Lions, J.-M. Morel, Axioms and fundamental equations in image processing, *Archive for Rational Mechanics and Analysis*, Vol. 123, 199–257, 1993.
  (established many theoretical results)
- S. Osher, J. A. Sethian, Fronts propagating with curvature-dependent speed: Algorithms based on Hamilton–Jacobi formulations, *Journal of Computational Physics*, Vol. 79, 12–49, 1988.
  (classical reference for level set methods)
- S. Osher and R. P. Fedkiw: *Level Set Methods and Dynamic Implicit Surfaces*. Springer, New York, 2002.
  (covers numerics as well as applications to image analysis and computational physics)
- J. A. Sethian: *Level Set Methods and Fast Marching Methods*, Cambridge University Press, Cambridge, 1999.
  (monograph on numerical methods and various applications of level set techniques, many of them beyond image processing and computer vision; references not always historically correct)
- M. Breuß, J. Weickert: A shock-capturing algorithm for the differential equations of dilation and erosion. *Journal of Mathematical Imaging and Vision*, Vol. 25, 187–201, 2006.
  (`http://www.mia.uni-saarland.de/Publications/breuss-pp153.pdf`)
  (FCT schemes)

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Assignment P5 – Programming

You can download the file Ex05.tar from the web page

`http://www.mia.uni-saarland.de/Teaching/dic08.shtml`

To unpack these data, use `tar xvf Ex05.tar`.

1. Supplement the ANSI C programme `optic.c` with the missing code such that it becomes a modified explicit scheme for flow-driven isotropic optic flow calculations with the Charbonnier diffusivity
$$g(s^2) \;=\; \Psi'(s^2) \;=\; \frac{1}{\sqrt{1+s^2/\lambda^2}}.$$
You can use it for calculating the optic flow between the images `pig1.pgm` and `pig2.pgm`. Experiment with the parameters α, λ and the number of iterations. What is their meaning ?

2. Modify the programme in such a way that homogeneous regularisation is used instead of the flow-driven isotropic smoothness term. Compare the results of the two regularisers.

Assignment P5 (2)

For assessment:
Use `tar -cvzf P5_`*yourname*`.tgz` *file1 file2 ...* to pack the following files into an archive:

- the supplemented file `optic.c`
- two results for different α, different λ and different number of iterations (six results in total)
- two results for the homgeneous regulariser
- a short `README` file stating the parameters used for the optic flow computation.
- a description of the meaning of the different parameters.

Deadline for electronic submission: Friday, June 27, 10 am (before the lecture).