

Lecture 14: Vector- and Matrix-Valued Images

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Motivation

Motivation

- ◆ So far, we have been focusing on diffusion and variational methods for scalar-valued images (greyscale images).
- ◆ However, vector- and matrix-valued data sets do also arise in a number of applications.
- ◆ Let us now generalise our approaches to these settings.
- ◆ We start by briefly reviewing the scalar-valued framework.

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Scalar-Valued Diffusion Filtering

◆ **Homogeneous Diffusion** (Iijima 1962)

$$\partial_t u = \Delta u$$

- noisy image f as initial condition for $t = 0$
- homogeneous: uses diffusivity that is not space-variant
- good for denoising, but blurs across edges

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◆ **Isotropic Nonlinear Diffusion** (Perona/Malik 1990, Catté et al. 1992)

$$\partial_t u = \operatorname{div} (g(|\nabla u_\sigma|^2) \nabla u)$$

- small diffusivity g at edges where $|\nabla u_\sigma|^2$ is large
- decreases diffusion at edges of the evolving image u
- preserves edges, but keeps also noise at edges for a long time
- allows forward–backward diffusion for rapidly decreasing diffusivities
- in this case Gaussian presmoothing is useful for increasing robustness w.r.t. to noise

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Scalar-Valued Diffusion Filtering (3)

◆ Edge-Enhancing Anisotropic Nonlinear Diffusion (W. 1994)

$$\partial_t u = \operatorname{div} (g(\nabla u_\sigma \nabla u_\sigma^\top) \nabla u)$$

- decreases diffusion across edges of the evolving image u
- diffusion tensor allows anisotropic behaviour
- preserves edges and removes noise by smoothing along edges
- tends to the isotropic Perona–Malik model for $\sigma \rightarrow 0$
(Lecture 11, Assignment C3, Problem 1)

◆ Coherence-Enhancing Anisotropic Nonlinear Diffusion (W. 1998)

$$\partial_t u = \operatorname{div} (D(J_\rho(\nabla u_\sigma)) \nabla u)$$

- allows to smooth along flow-like structures

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Scalar-Valued Diffusion Filtering (4)

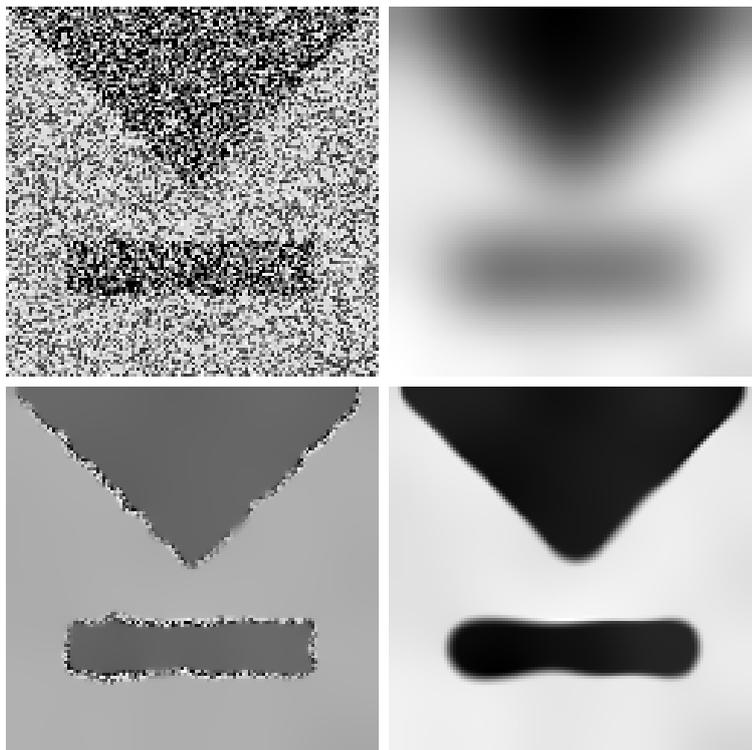


Image enhancement with diffusion filters. (a) **Top left:** Test image. (b) **Top right:** Homogeneous diffusion. (c) **Bottom left:** Isotropic nonlinear diffusion. (d) **Bottom right:** Edge-enhancing anisotropic nonlinear diffusion.

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Scalar-Valued Variational Methods

◆ **Homogeneous Smoothing** (Whittaker 1923, Tikhonov 1963)

- Consider the quadratic functional (Tikhonov regularisation)

$$E(u) = \int_{\Omega} ((u - f)^2 + \alpha |\nabla u|^2) dx$$

- Its Euler–Lagrange equation approximates homogeneous diffusion:

$$\frac{u - f}{\alpha} = \Delta u$$

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◆ **Isotropic Nonlinear Smoothing**

$$E(u) = \int_{\Omega} ((u - f)^2 + \alpha \Psi(|\nabla u|^2)) dx$$

- Euler–Lagrange equation approximates isotropic nonlinear diffusion:

$$\frac{u - f}{\alpha} = \operatorname{div} (\Psi'(|\nabla u|^2) \nabla u)$$

- well-posed for penalisers $\Psi(s^2)$ that are convex in s
- correspond to the forward diffusion case (Lecture 10, Assignment T3, Problem 4)

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Scalar-Valued Variational Methods (3)

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Remarks:

- ◆ The results for the regularisation methods look similar to the corresponding diffusion methods.
- ◆ So far no energy functionals have been found that approximate nonlinear diffusion filters with Gaussian regularisation.
- ◆ Since in the *scalar-valued (!)* setting, anisotropic diffusion filters require Gaussian regularisation, no energy functionals for scalar-valued nonlinear anisotropic smoothing are available yet.

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Vector-Valued Diffusion Filtering (1)

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Vector-Valued Diffusion Filtering

Where do vector-valued images arise?

- ◆ colour images (e.g. RGB)
- ◆ multi-spectral satellite images
- ◆ multi-spin echo MR images

Why not diffuse each channel separately?

- ◆ works well for homogeneous diffusion
- ◆ for space-variant filters, structures may be formed at different locations for the different channels
⇒ coupling between channels desirable

assumption: channels have comparable intensity ranges

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Vector-Valued Diffusion Filtering (2)

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◆ Homogeneous Diffusion

$$\partial_t u_i = \Delta u_i \quad (i = 1, \dots, m)$$

- no coupling between the channels

◆ Isotropic Nonlinear Diffusion (Gerig et al. 1992)

$$\partial_t u_i = \operatorname{div} \left(g \left(\sum_{k=1}^m |\nabla u_{k,\sigma}|^2 \right) \nabla u_i \right) \quad (i = 1, \dots, m)$$

- reduces diffusion at edges of the evolving image u
- channels coupled via a joint diffusivity

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Vector-Valued Diffusion Filtering (3)

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◆ Edge-Enhancing Anisotropic Nonlinear Diffusion (W. 1994)

$$\partial_t u_i = \operatorname{div} \left(g \left(\sum_{k=1}^m \nabla u_{k,\sigma} \nabla u_{k,\sigma}^\top \right) \nabla u_i \right) \quad (i = 1, \dots, m)$$

- coupling via a joint diffusion tensor
- may behave also anisotropically for $\sigma \rightarrow 0$, since ∇u_i is not necessarily parallel to one eigenvector of $g \left(\sum_{k=1}^m \nabla u_k \nabla u_k^\top \right)$

◆ Coherence-Enhancing Anisotropic Nonlinear Diffusion (W. 1999)

$$\partial_t u_i = \operatorname{div} \left(D \left(\sum_{k=1}^m J_\rho(\nabla u_{k,\sigma}) \right) \nabla u_i \right) \quad (i = 1, \dots, m)$$

- coupling via a joint diffusion tensor

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Vector-Valued Diffusion Filtering (4)

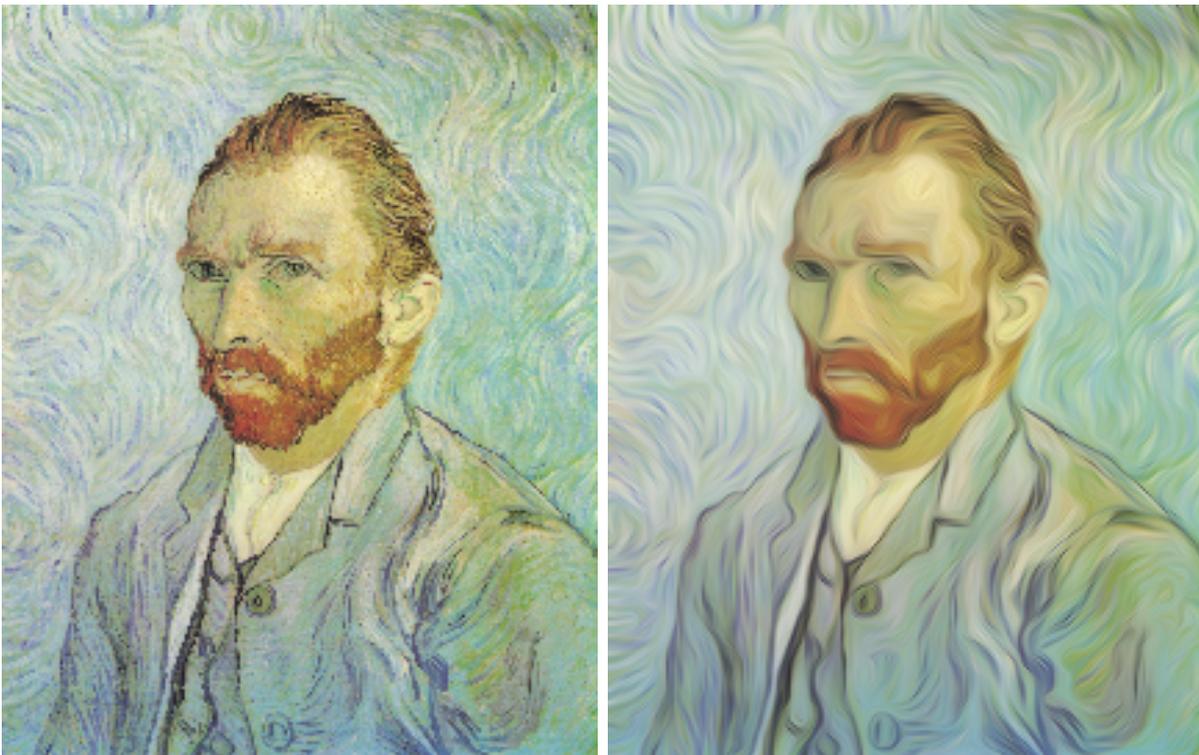


(a) **Top left:** Noisy original image. (b) **Top right:** Homogeneous diffusion. (c) **Bottom left:** Isotropic nonlinear diffusion. (d) **Bottom right:** Edge-enhancing anisotropic nonlinear diffusion.

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Vector-Valued Diffusion Filtering (5)

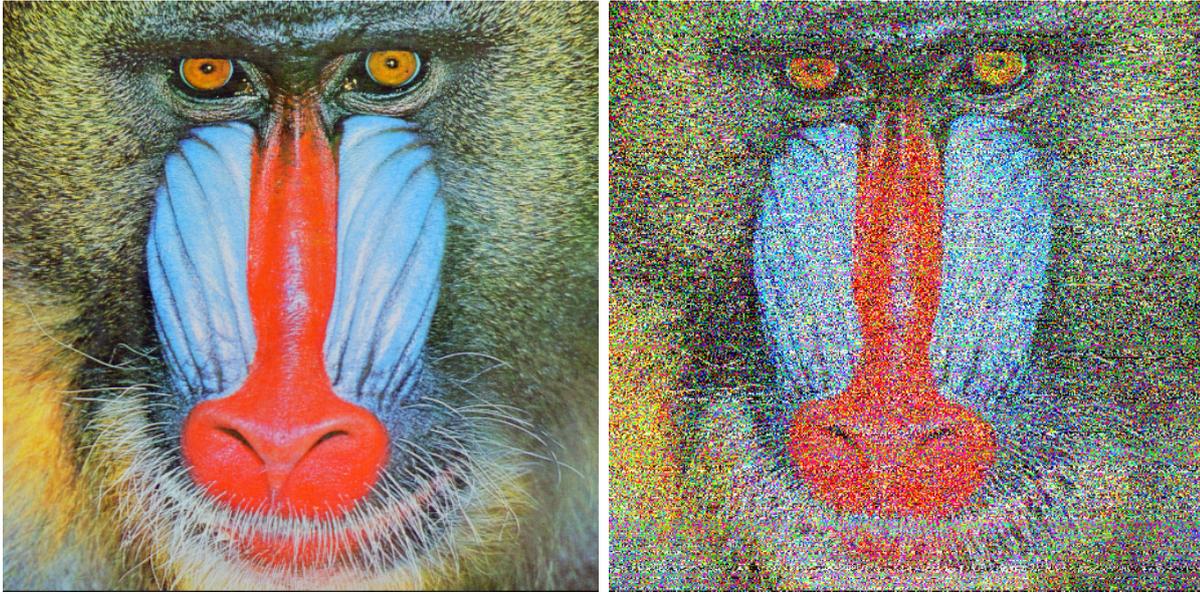


(a) **Left:** Self portrait by van Gogh (Saint-Rémy, 1889; Paris, Musée d'Orsay). (b) **Right:** Coherence-enhancing nonlinear diffusion filtering.

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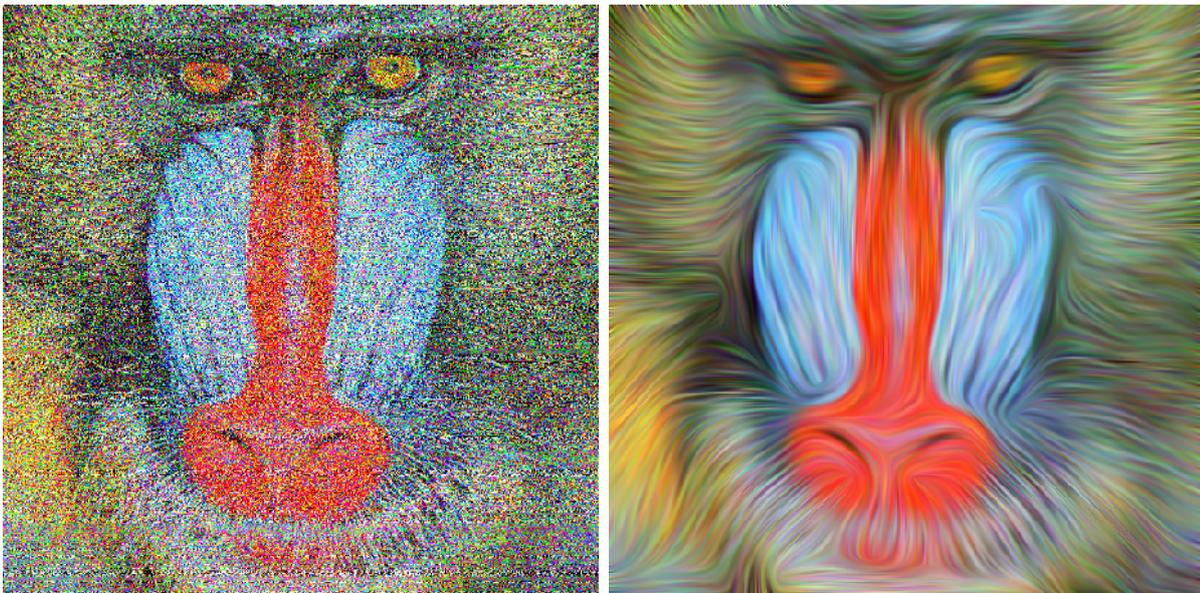
Vector-Valued Diffusion Filtering (6)



(a) **Left:** Mandrill, 512×512 pixels. (b) **Right:** Severely degraded by Gaussian noise.

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Vector-Valued Diffusion Filtering (7)



(a) **Left:** Noisy mandrill image. (b) **Right:** Filtered by coherence-enhancing anisotropic diffusion.

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Vector-Valued Variational Methods

Homogeneous Smoothing

$$E_f(\mathbf{u}) := \int_{\Omega} \left(\|\mathbf{u} - \mathbf{f}\|_2^2 + \alpha \left(\sum_k |\nabla u_k|^2 \right) \right) dx$$

- ◆ $\|\mathbf{u}\|_2$ denotes the Euclidean norm of the vector \mathbf{u} :

$$\|\mathbf{u}\|_2 := \sqrt{\sum_{i=1}^m u_i^2}$$

- ◆ creates the time-discrete homogeneous filter

$$\frac{u_i - f_i}{\alpha} = \Delta u_i \quad \forall i$$

- ◆ no coupling between the channels

Isotropic Nonlinear Smoothing

$$E_f(\mathbf{u}) := \int_{\Omega} \left(\|\mathbf{u} - \mathbf{f}\|_2^2 + \alpha \Psi \left(\sum_k |\nabla u_k|^2 \right) \right) dx$$

creates the time-discrete isotropic diffusion filter

$$\frac{u_i - f_i}{\alpha} = \operatorname{div} \left(\Psi' \left(\sum_k |\nabla u_k|^2 \right) \nabla u_i \right) \quad \forall i$$

Anisotropic Nonlinear Smoothing (W./Schnörr 2001)

$$E_f(\mathbf{u}) := \int_{\Omega} \left(\|\mathbf{u} - \mathbf{f}\|_2^2 + \alpha \operatorname{tr} \Psi \left(\sum_k \nabla u_k \nabla u_k^\top \right) \right) dx$$

gives the desired time-discrete edge-enhancing anisotropic diffusion filter

$$\frac{u_i - f_i}{\alpha} = \operatorname{div} \left(\Psi' \left(\sum_k \nabla u_k \nabla u_k^\top \right) \nabla u_i \right) \quad \forall i$$

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Matrix-Valued Diffusion Filtering

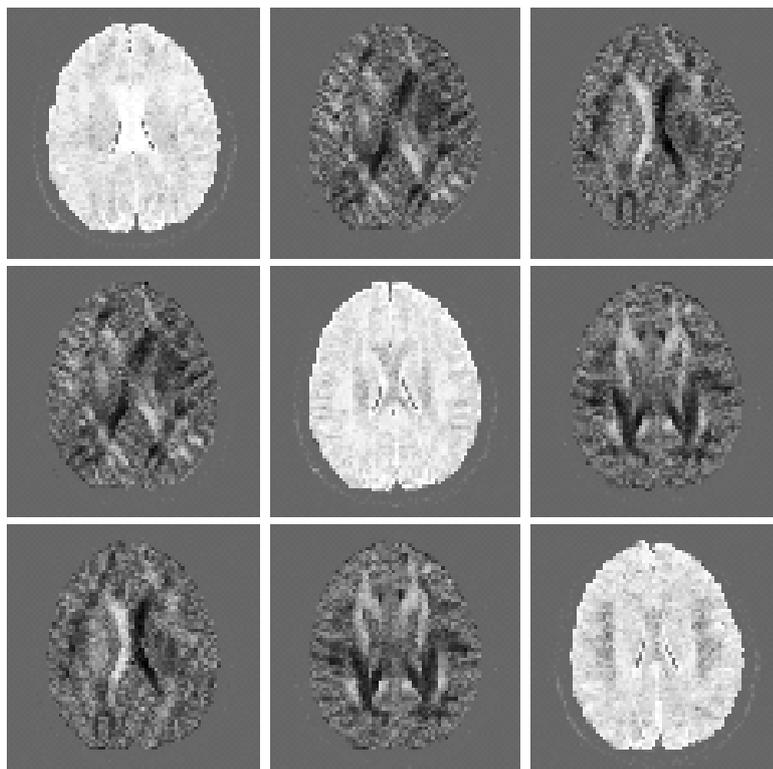
Examples for Matrix-Valued Images

- ◆ *Diffusion Tensor Magnetic Resonance Imaging (DT-MRI)*
 - measure 3×3 positive definite matrix in every voxel
 - quantifies diffusion properties of water molecules in tissues
 - useful for stroke diagnosis and brain connectivity studies
- ◆ *Other Physical Measurements of Anisotropic Behaviour*
 - e.g. stress-strain relationships, inertia tensors, permittivity tensors
- ◆ *Matrix-Valued Image Features*
 - e.g. structure tensor (cf. Lecture 7)

Additional Constraint

- ◆ often the matrices are symmetric and positive semidefinite
- ◆ positive semidefinite fields should remain positive semidefinite

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Nine coefficients of a diffusion tensor MR image. Only six of them are different, since the diffusion matrix is symmetric. From Weinstein et al. (1999).

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Matrix-Valued Diffusion Filtering (3)



Similar formal requirements as for vector-valued diffusion:

- ◆ coupling should synchronise evolution in all channels
- ◆ rotationally invariant filtering

guaranteed by use of tensor products

◆ Homogeneous Diffusion

$$\partial_t u_{i,j} = \Delta u_{i,j} \quad \forall i, j$$

◆ Isotropic Nonlinear Diffusion (Tschumperlé/Deriche 2001)

$$\partial_t u_{i,j} = \operatorname{div} \left(g \left(\sum_{k,l} |\nabla u_{k,l,\sigma}|^2 \right) \nabla u_{i,j} \right) \quad \forall i, j$$

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Matrix-Valued Diffusion Filtering (3)



◆ Edge-Enhancing Anisotropic Nonlinear Diffusion (W./Brox 2002)

$$\partial_t u_{i,j} = \operatorname{div} \left(g \left(\sum_{k,l} \nabla u_{k,l,\sigma} \nabla u_{k,l,\sigma}^\top \right) \nabla u_{i,j} \right) \quad \forall i, j$$

◆ Coherence-Enhancing Anisotropic Nonlinear Diffusion

$$\partial_t u_{i,j} = \operatorname{div} \left(D \left(\sum_{k,l} J_\rho(\nabla u_{k,l,\sigma}) \right) \nabla u_{i,j} \right) \quad \forall i, j$$

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Matrix-Valued Diffusion Filtering (4)

Preservation of Positive Semidefiniteness

Discrete diffusion filtering in matrix–vector notation:

$$\begin{aligned} \mathbf{u}_{i,j}^0 &= \mathbf{f}_{i,j} && \forall i, j \\ \mathbf{u}_{i,j}^{k+1} &= Q(\mathbf{u}_{1,1}^k, \dots, \mathbf{u}_{m,m}^k) \mathbf{u}_{i,j}^k && \forall i, j, \quad \forall k \geq 0 \end{aligned}$$

- ◆ matrix Q has unit row sums and nonnegative entries
- ◆ identical matrix Q for all channels

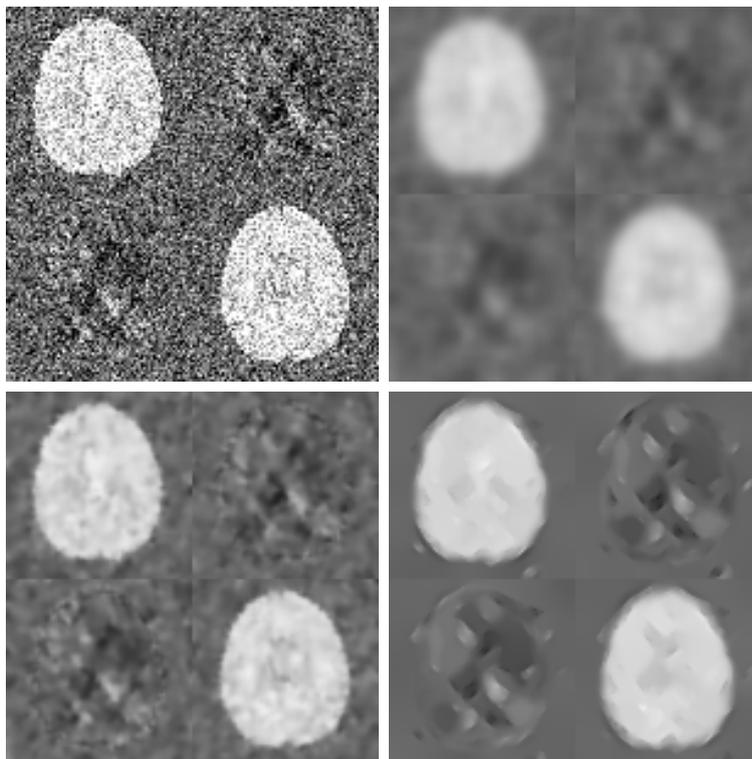
Consequences:

- ◆ discrete diffusion schemes perform convex combinations of matrices
- ◆ positive semidefinite fields remain positive semidefinite

Proof can be extended to the continuous setting (Brox et al. 2006).

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Matrix-Valued Diffusion Filtering (5)



(a) **Top left:** 2D matrix-valued image degraded by Gaussian noise. The four quadrants depict the (1,1), (1,2), (2,1) and (2,2) channels. (b) **Top right:** Homogeneous diffusion filtering. (c) **Bottom left:** Isotropic nonlinear diffusion. (d) **Bottom right:** Edge-enhancing anisotropic nonlinear diffusion.

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Matrix-Valued Variational Methods

Homogeneous Smoothing

$$E_f(\mathbf{u}) := \int_{\Omega} \left(\|\mathbf{u} - \mathbf{f}\|_2^2 + \alpha \sum_{k,l} |\nabla u_{k,l}|^2 \right) dx$$

- ◆ $\|\mathbf{u}\|_2$ denotes the Frobenius norm of the matrix \mathbf{u} :

$$\|\mathbf{u}\|_2 := \sqrt{\sum_{i,j} u_{i,j}^2}$$

- ◆ creates the time-discrete homogeneous filter

$$\frac{u_{i,j} - f_{i,j}}{\alpha} = \Delta u_{i,j} \quad \forall i, j$$

- ◆ no coupling between the channels

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Isotropic Nonlinear Regularisation (Tschumperlé/Deriche 2001)

$$E_f(\mathbf{u}) := \int_{\Omega} \left(\|\mathbf{u} - \mathbf{f}\|_2^2 + \alpha \Psi \left(\sum_{k,l} |\nabla u_{k,l}|^2 \right) \right) dx$$

creates the time-discrete isotropic diffusion filter

$$\frac{u_{i,j} - f_{i,j}}{\alpha} = \operatorname{div} \left(\Psi' \left(\sum_{k,l} |\nabla u_{k,l}|^2 \right) \nabla u_{i,j} \right) \quad \forall i, j$$

Anisotropic Nonlinear Regularisation (W./Brox 2002)

$$E_f(\mathbf{u}) := \int_{\Omega} \left(\|\mathbf{u} - \mathbf{f}\|_2^2 + \alpha \operatorname{tr} \Psi \left(\sum_{k,l} \nabla u_{k,l} \nabla u_{k,l}^\top \right) \right) dx$$

yields the time-discrete edge-enhancing anisotropic diffusion filter

$$\frac{u_{i,j} - f_{i,j}}{\alpha} = \operatorname{div} \left(\Psi' \left(\sum_{k,l} \nabla u_{k,l} \nabla u_{k,l}^\top \right) \nabla u_{i,j} \right) \quad \forall i, j$$

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Summary

- ◆ Most ideas from scalar-valued diffusion filtering can be extended in a straightforward way to vector- and matrix-valued images.
- ◆ In this case one should use joint diffusivities / diffusion tensors for all channels.
- ◆ For most diffusion filters it is also possible to find corresponding variational methods.

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(*vector-valued EED*)

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(proof of preservation of positive semidefiniteness in the continuous case)

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Assignment T4 (1)



Assignment T4 – Theoretical Home Work

Problem 1 (Continuous Variational Regularisation) (3 points)

Let the following energy functional be defined for sufficiently smooth functions u on the rectangle Ω :

$$E_f(u) := \int_{\Omega} \left(\Psi_1 \left((u - f)^2 \right) + \alpha \Psi_2 \left(|\nabla u|^2 \right) \right) dx$$

with $\Psi_1(s^2) := \sqrt{s^2 + \varepsilon^2}$, $\Psi_2 := \lambda^2 \log(1 + s^2/\lambda^2)$, and parameters $\alpha, \varepsilon, \lambda > 0$.

Derive the Euler-Lagrange equation and its corresponding gradient descent that each u of the functional E_f satisfies necessarily.

Problem 2 (Rotation Invariance of Energy Functionals) (3 points)

We consider energy functionals of the type

$$E_f(u) := \int_{\Omega} \left((u - f)^2 + \alpha \Psi(u_x, u_y) \right) dx dy \tag{1}$$

where the smoothness term depends on the two partial derivatives of u of first order.

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Assignment T4 (2)

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The energy functional E is called *rotationally invariant* if it does not change its value when the input and filtered images are subject to the same rotation operator R , i. e. if we have

$$E_{Rf}(Ru) = E_f(u)$$

for all images f, u and all rotations R . For energy functionals of the form (1), this is equivalent to the requirement that the penaliser Ψ is rotationally invariant, i. e. one has $\Psi(u_\xi, u_\eta) = \Psi(u_x, u_y)$ whenever ξ, η are coordinates in a rotated orthonormal frame,

$$\begin{aligned} \xi &= x \cos \vartheta + y \sin \vartheta, \\ \eta &= -x \sin \vartheta + y \cos \vartheta \end{aligned}$$

with a constant angle ϑ .

For each of the following expressions for the smoothness penaliser:

◆ $\Psi(u_x, u_y) = |u_x| + |u_y|$

◆ $\Psi(u_x, u_y) = u_x^2 + u_y^2$

determine whether it is rotationally invariant or not.

Hint: Remember that $\partial_z u = \nabla u^\top \mathbf{z}$ for any unit vector $\mathbf{z} = (z_x, z_y)^\top$.

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Assignment T4 (3)

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Problem 3 (Discrete Energy Minimisation)

(3 points)

Consider the following discrete energy function E_f for an N -pixel signal (u_1, \dots, u_N) :

$$E_f(u_1, \dots, u_N) = \sum_{k=1}^N (u_k - f_k)^2 + \alpha \sum_{k=1}^{N-1} (u_{k+1} - u_k)^2 + \beta \sum_{k=2}^{N-1} (u_{k+1} - 2u_k + u_{k-1}) .$$

- (a) Derive by differentiation the system of equations that a minimiser (u_1^*, \dots, u_N^*) of E_f satisfies necessarily. Pay special attention to the correct treatment of pixels near the boundary.
- (b) The function E_f can be understood as discretisation of a continuous energy functional containing smoothness terms with the first and second derivatives of the smooth image u . Its corresponding Euler-Lagrange equation would therefore contain fourth-order derivatives. Identify in your system from (a) the expressions that are discretisations of fourth order derivatives.

Problem 4 (Half-Quadratic Regularisation)

(3 points)

Derive the half-quadratic regularisation formulation corresponding to the (nonconvex) energy function

$$E_f(u) := \int_{\Omega} \left((u - f)^2 + \alpha \cdot 2\lambda^2 \left(1 - \exp \left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \right) \right) dx \quad (\alpha, \lambda > 0).$$

Deadline for submission: Friday, June 6, 10 am (before the lecture).

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