

Lecture 8:

Nonlinear Anisotropic Diffusion II: Theoretical and Numerical Aspects

Contents

1. Continuous Well-Posedness and Scale-Space Theory
2. Semidiscrete and Discrete Theory
3. Standard Semidiscretisations
4. Nonnegative Semidiscretisations on a (3×3) -Stencil
5. Nonnegative Semidiscretisations in General
6. Alternatives to Nonnegative Semidiscretisations

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Continuous Well-Posedness and Scale-Space Theory (1)

Continuous Well-Posedness and Scale-Space Theory

Motivation

All before mentioned diffusion filters can be cast in the form

$$\partial_t u = \operatorname{div}(D(J_\rho(\nabla u_\sigma)) \nabla u).$$

- (a) *Coherence-enhancing anisotropic diffusion:* $D = D(J_\rho(\nabla u_\sigma)).$
- (b) *Edge-enhancing anisotropic diffusion:* $D = g(\nabla u_\sigma \nabla u_\sigma^\top) = g(J_0(\nabla u_\sigma)).$
- (c) *Isotropic nonlinear diffusion:* $D = g(|\nabla u_\sigma|^2) I = g(\operatorname{tr}(J_0(\nabla u_\sigma))) I.$
- (d) *Linear diffusion:* $D = I.$

Is there a general well-posedness and scale-space framework?

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Assumptions on the Anisotropic Filter Class

Let f be bounded, $\rho \geq 0$, $\sigma > 0$ and consider

$$\begin{aligned} \partial_t u &= \operatorname{div} (D(J_\rho(\nabla u_\sigma)) \nabla u) && \text{on } \Omega \times (0, T], \\ u(\mathbf{x}, 0) &= f(\mathbf{x}) && \text{on } \Omega, \\ \mathbf{n}^\top D(J_\rho(\nabla u_\sigma)) \nabla u &= 0 && \text{on } \partial\Omega \times (0, T], \end{aligned}$$

where the diffusion tensor $D = (d_{i,j})$ satisfies

◆ **(C1) Smoothness:**

$$D \in C^\infty(\mathbb{R}^{2 \times 2}, \mathbb{R}^{2 \times 2}).$$

◆ **(C2) Symmetry:**

$$d_{1,2}(J) = d_{2,1}(J) \text{ for all symmetric matrices } J \in \mathbb{R}^{2 \times 2}.$$

◆ **(C3) Uniform Positive Definiteness:**

For all bounded functions $\mathbf{w} : \Omega \rightarrow \mathbb{R}^2$ with $|\mathbf{w}(x)| \leq K$ on $\bar{\Omega}$, there exists a positive lower bound $\nu(K)$ for the eigenvalues of $D(J_\rho(\mathbf{w}))$.

Results

(a) **Well-Posedness and Regularity**

- Existence of a unique (distributional) solution $u(\mathbf{x}, t)$
- Smoothness: $u \in C^\infty(\bar{\Omega} \times (0, \infty))$
- $u(\cdot, t)$ depends continuously on f w. r. t. $\|\cdot\|_{L^2(\Omega)}$.

(b) **Average Grey Level Invariance**

$$\frac{1}{|\Omega|} \int_{\Omega} u(\mathbf{x}, t) \, d\mathbf{x} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{x}) \, d\mathbf{x} =: \mu.$$

(c) **Maximum–Minimum Principle**

$$\inf f \leq u(\mathbf{x}, t) \leq \sup f \quad \forall \mathbf{x} \in \bar{\Omega}, \forall t > 0$$

(d) **Lyapunov Functionals**

$$V(t) := \Phi(u(t)) := \int_{\Omega} r(u(\mathbf{x}, t)) dx$$

is a Lyapunov function for all convex $r \in C^2$:
 $V(t)$ is decreasing and bounded from below.

(e) **Convergence to a Constant Steady-State**

$u(\mathbf{x}, t)$ converges to the average grey level μ
 (Convergence in $L^p(\Omega)$, $1 \leq p \leq \infty$).

Thus, we have the same results as in the isotropic nonlinear case (Lecture 4).

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Semidiscrete and Discrete Theory

Same (semi-)discrete theory as in the isotropic nonlinear case (Lecture 5).

Five Requirements in Each Framework:

requirement	continuous $\partial_t u = \text{div}(D\nabla u)$ $u(\mathbf{x}, 0) = f(\mathbf{x})$	semidiscrete $\frac{d\mathbf{u}}{dt} = A(\mathbf{u})\mathbf{u}$ $\mathbf{u}(0) = \mathbf{f}$	discrete $\mathbf{u}^{k+1} = Q(\mathbf{u}^k)\mathbf{u}^k$ $\mathbf{u}^0 = \mathbf{f}$
smoothness	$D \in C^\infty$	A Lipschitz-continuous	Q continuous
symmetry	D symmetric	A symmetric	Q symmetric
conservation	divergence form; $\mathbf{n}^\top D\nabla u = 0$	$\sum_i a_{i,j} = 0$	$\sum_i q_{i,j} = 1$
nonnegativity	positive semidefinite	nonnegative off-diagonals	nonnegative elements
connectivity	uniformly positive definite	A irreducible	Q irreducible; pos. diagonal

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Five Similar Results:

- (a) well-posedness
- (b) average grey level invariance
- (c) maximum–minimum principle
- (d) Lyapunov functionals / functions / sequences:
decreasing energy, variance, increasing entropy
- (e) convergence to a constant steady-state

Do typical discretisations satisfy these properties ?

- ◆ We shall see that the space discretisation can create problems.
- ◆ The time discretisation does not give new aspects compared to isotropic nonlinear diffusion.
- ◆ Therefore it is sufficient to study semidiscretisations that are discrete in space and continuous in time.

Standard Semidiscretisation (1)**Standard Semidiscretisations**

Let $D = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. Then

$$\begin{aligned} \partial_t u &= \operatorname{div}(D \nabla u) = \operatorname{div} \begin{pmatrix} a \partial_x u + b \partial_y u \\ b \partial_x u + c \partial_y u \end{pmatrix} \\ &= \partial_x(a \partial_x u) + \partial_x(b \partial_y u) + \partial_y(b \partial_x u) + \partial_y(c \partial_y u) \end{aligned}$$

novelty: mixed terms $\partial_x(b \partial_y u) + \partial_y(b \partial_x u)$

standard approximations by central differences:

$$\partial_x(b \partial_y u) \approx \frac{1}{2h_1} \left(b_{i+1,j} \frac{u_{i+1,j+1} - u_{i+1,j-1}}{2h_2} - b_{i-1,j} \frac{u_{i-1,j+1} - u_{i-1,j-1}}{2h_2} \right)$$

$$\partial_y(b \partial_x u) \approx \frac{1}{2h_2} \left(b_{i,j+1} \frac{u_{i+1,j+1} - u_{i-1,j+1}}{2h_1} - b_{i,j-1} \frac{u_{i+1,j-1} - u_{i-1,j-1}}{2h_1} \right)$$

Standard Semidiscretisation (2)

stencil weights for standard approximation of $\operatorname{div} \left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} \nabla u \right)$:

$\frac{-b_{i-1,j} - b_{i,j+1}}{4h_1h_2}$	$\frac{c_{i,j+1} + c_{i,j}}{2h_2^2}$	$\frac{b_{i+1,j} + b_{i,j+1}}{4h_1h_2}$
$\frac{a_{i-1,j} + a_{i,j}}{2h_1^2}$	$-\frac{a_{i-1,j} + 2a_{i,j} + a_{i+1,j}}{2h_1^2}$ $-\frac{c_{i,j-1} + 2c_{i,j} + c_{i,j+1}}{2h_2^2}$	$\frac{a_{i+1,j} + a_{i,j}}{2h_1^2}$
$\frac{b_{i-1,j} + b_{i,j-1}}{4h_1h_2}$	$\frac{c_{i,j-1} + c_{i,j}}{2h_2^2}$	$\frac{-b_{i+1,j} - b_{i,j-1}}{4h_1h_2}$

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Standard Semidiscretisation (3)

Is This Discretisation Nonnegative ?

◆ If $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ is positive semidefinite, then $a \geq 0$ and $c \geq 0$.
However, b may have arbitrary sign.

◆ sign pattern of the stencil:

?	+	?
+	-	+
?	+	?

◆ Negative ? destroy nonnegativity condition (S4) for off-diagonals of A in

$$\frac{d\mathbf{u}}{dt} = A(\mathbf{u}) \mathbf{u}$$

◆ Thus, the semidiscrete scale-space framework is not applicable for the standard semidiscretisation.

Are there semidiscretisations that do not violate (S4)?

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Nonnegative Semidiscretisation on a (3×3) -Stencil

Main Result:

Let $D \in \mathbb{R}^{2 \times 2}$ be symmetric positive definite.

A nonnegative second-order discretisation of $\text{div}(D \nabla u)$ on a (3×3) -stencil with $h_1 = h_2$ can be guaranteed, if the spectral condition number $\text{cond}(D)$ satisfies

$$\text{cond}(D) \leq \frac{1 + \frac{1}{2}\sqrt{2}}{1 - \frac{1}{2}\sqrt{2}} = 3 + 2\sqrt{2} \approx 5.8284.$$

This bound cannot be improved!

Consequence:

To guarantee the nonnegativity condition (S4) on a (3×3) -stencil, the ratio between the eigenvalues of D may not exceed 5.8284. This limits the anisotropy!

Here is a (3×3) -stencil that is nonnegative (outside the centre) for $\text{cond}(D) \leq 5.8284$:

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$\frac{ b_{i-1,j+1} - b_{i-1,j+1}}{4h_1h_2} + \frac{ b_{i,j} - b_{i,j}}{4h_1h_2}$	$\frac{c_{i,j+1} + c_{i,j}}{2h_2^2} - \frac{ b_{i,j+1} + b_{i,j} }{2h_1h_2}$	$\frac{ b_{i+1,j+1} + b_{i+1,j+1}}{4h_1h_2} + \frac{ b_{i,j} + b_{i,j}}{4h_1h_2}$
$\frac{a_{i-1,j} + a_{i,j}}{2h_1^2} - \frac{ b_{i-1,j} + b_{i,j} }{2h_1h_2}$	$-\frac{a_{i-1,j} + 2a_{i,j} + a_{i+1,j}}{2h_1^2} - \frac{ b_{i-1,j+1} - b_{i-1,j+1} + b_{i+1,j+1} + b_{i+1,j+1}}{4h_1h_2} - \frac{ b_{i-1,j-1} + b_{i-1,j-1} + b_{i+1,j-1} - b_{i+1,j-1}}{4h_1h_2} + \frac{ b_{i-1,j} + b_{i+1,j} + b_{i,j-1} + b_{i,j+1} + 2 b_{i,j} }{2h_1h_2} - \frac{c_{i,j-1} + 2c_{i,j} + c_{i,j+1}}{2h_2^2}$	$\frac{a_{i+1,j} + a_{i,j}}{2h_1^2} - \frac{ b_{i+1,j} + b_{i,j} }{2h_1h_2}$
$\frac{ b_{i-1,j-1} + b_{i-1,j-1}}{4h_1h_2} + \frac{ b_{i,j} + b_{i,j}}{4h_1h_2}$	$\frac{c_{i,j-1} + c_{i,j}}{2h_2^2} - \frac{ b_{i,j-1} + b_{i,j} }{2h_1h_2}$	$\frac{ b_{i+1,j-1} - b_{i+1,j-1}}{4h_1h_2} + \frac{ b_{i,j} - b_{i,j}}{4h_1h_2}$

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Nonnegative Semidiscretisations in General

Is it possible to use larger condition numbers on a larger stencil?

Yes!

For some arbitrary condition number, is there a suitable stencil?

Yes! For every condition number $\text{cond}(D)$ there exists some $m(\text{cond}(D)) \in \mathbb{N}$ such that $\text{div}(D \nabla u)$ has a nonnegative second-order discretisation on a $(2m+1) \times (2m+1)$ -stencil.

Is the proof constructive?

Yes. It shows how to choose 3 stencil directions which are sufficient: x and y axis, and a third direction close to one eigenvector of D .

So, what's the problem?

- ◆ discretisation may not use smallest possible stencil size
- ◆ large stencils are computationally expensive and can introduce blurring effects

Alternatives to Nonnegative Semidiscretisations

- ◆ Nonnegativity ensures a very strict stability result, namely a maximum–minimum principle. This implies so-called **stability in the maximum norm**.
- ◆ A numerical scheme $u^{k+1} = Q(u^k) u^k$ is **stable in the p -norm**, if $\|u^{k+1}\|_p \leq \|u^k\|_p$. The **p -norm** in \mathbb{R}^N is defined as

$$\|u\|_p := \left(\sum_{i=1}^N |u_i|^p \right)^{1/p} \quad (1 \leq p < \infty)$$

$$\|u\|_\infty := \max_i |u_i|$$

- ◆ Schemes that violate nonnegativity may still be useful. They may satisfy weaker forms of stability such as stability in the 2-norm (Euclidean norm).
- ◆ One can establish stability in the 2-norm for standard or nonnegativity space discretisations in combination with explicit or semi-implicit time discretisations. In the explicit case, time step restrictions apply.
- ◆ Such schemes with stability in the 2-norm may violate a maximum–minimum principle, but their over- and undershoots remain bounded.

Summary

- ◆ general continuous well-posedness and scale-space theory:
covers linear, nonlinear isotropic and anisotropic diffusion filters
(exception: unregularised Perona-Malik filter)

- ◆ anisotropic filters have tedious mixed derivatives:

$$\partial_x(b \partial_y u) + \partial_y(b \partial_x u)$$

- ◆ Standard space discretisations violate nonnegativity (S4).
- ◆ may still lead to simple and practically useful schemes that satisfy weaker forms of stability (such as stability in the 2-norm)
- ◆ nonnegative discretisations on (3×3) -stencil possible if

$$\text{cond}(D) \leq 3 + 2\sqrt{2} \approx 5.8284.$$

- ◆ some open questions for larger condition numbers

Reference

- ◆ J. Weickert, *Anisotropic Diffusion in Image Processing*, Teubner, Stuttgart, 1998.
(www.mia.uni-saarland.de/weickert/Papers/diss.ps.gz)
(Chapter 2 treats the continuous theory, and Chapters 3 and 4 are devoted to the semidiscrete and discrete theory, respectively.)

Assignment P2 (1)



Assignment P2 – Programming

Download the file `Ex02.tar` from <http://www.mia.uni-saarland.de/Teaching/dic08.shtml> and unpack it by `tar xvf Ex02.tar`.

- (a) Implement the explicit finite difference scheme for the regularised isotropic nonlinear diffusion filter.
- ◆ The main programme is available as object code `ind.o`. It contains all necessary auxiliary routines except for the diffusion routine `isonondiff`.
 - ◆ Go to the file `isonondiff.c` and supplement it with the missing code for creating a 2D array `dc` with the diffusivities at each pixel. Use the diffusivity

$$g(|\nabla u_\sigma|^2) := \begin{cases} 1 & \text{if } |\nabla u_\sigma|^2 = 0, \\ 1 - \exp\left(\frac{-3.31488}{(|\nabla u_\sigma|/\lambda)^8}\right) & \text{if } |\nabla u_\sigma|^2 > 0 \end{cases}$$

and be sure that its argument is the Gaussian-smoothed image.

- (b) Compile the programme with `gcc -O2 -o ind ind.o isonondiff.c -lm` and run it on the image `wood.pgm`. It depicts a wood surface with a defect. Use the parameters $\lambda = 7$, $\sigma = 1$, $\tau = 0.2$, and apply 600 iterations.
- (c) Use the Perona-Malik diffusivity

$$g(|\nabla u_\sigma|^2) := \frac{1}{1 + |\nabla u_\sigma|^2/\lambda^2}$$

to denoise the image `dic08-noise.pgm`. Search for appropriate parameters.

Assignment P2 (2)



- (d) Try also the Charbonnier diffusivity

$$g(|\nabla u_\sigma|^2) = \frac{1}{\sqrt{1 + |\nabla u_\sigma|^2/\lambda^2}}$$

to denoise `dic08-noise.pgm`. What do you notice with respect to edge preservation and enhancement?

- (e) For the previous two denoising methods, use the programme `difference` from the first programming assignment to determine and compare the method noise of both approaches.

For assessment:

Use `tar cvzf P2_yourname.tgz file1 file2 ...` to pack the following files into an archive:

- ◆ the supplemented code for `isonondiff.c` for all diffusivities you have used,
- ◆ a representative image for both the defect detection and the two denoising problems,
- ◆ the corresponding method noise images.

Include in the archive also a short README file stating the parameters (number of iterations, time step size, λ , σ , diffusivity type) for each of the submitted images. Send the archive by e-mail to the address specified by your tutor.

Deadline for electronic submission: Friday, May 16, 10 am.

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