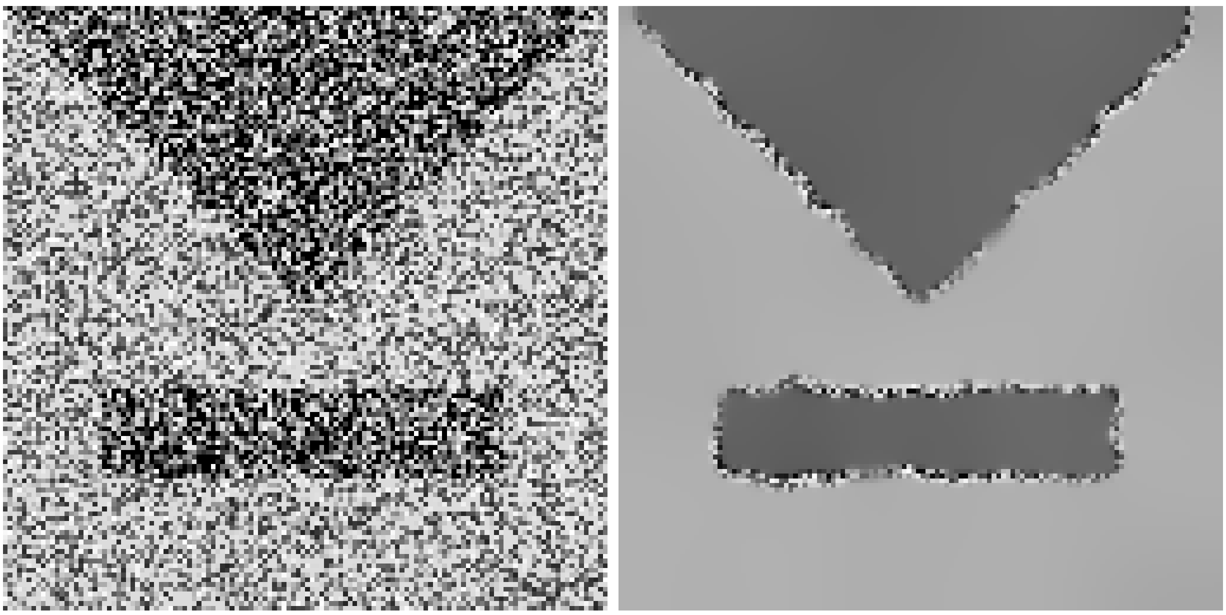
Lecture 7:

Nonlinear Anisotropic Diffusion I: Modelling

Contents

Why Anisotropic?



(a) **Left:** Test image, $\Omega = (0, 128)^2$. (b) **Right:** Isotropic nonlinear diffusion is not optimal when noise at edges is to be removed ($\lambda = 3.5$, $\sigma = 3$, $t = 80$).

Why Anisotropic? (2)

Fingerprint processing. (a) **Left:** Original fingerprint, $\Omega = (0, 256)^2$. (b) **Right:** Isotropic nonlinear diffusion cannot close interrupted lines ($\sigma = 1$, $\lambda = 4$, $t = 20$).

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

Why Anisotropic? (3)

Isotropic nonlinear diffusion filters

- can have difficulties with noisy edges
- cannot enhance coherent flow-like structures

Improved filters must take into account the *direction* of the local structure.

$\longrightarrow$ requires a diffusion tensor instead of scalar-valued diffusivity

$\longrightarrow$ local structure specifies its eigenvectors and eigenvalues

Two examples: 1. Noise elimination with edge enhancement

2. Enhancement of coherent structures

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

Edge-Enhancing Anisotropic Diffusion (EED)

Goals:

- smooth preferently within regions
- inhibit diffusion perpendicular to edges

Choice of the Diffusion Tensor

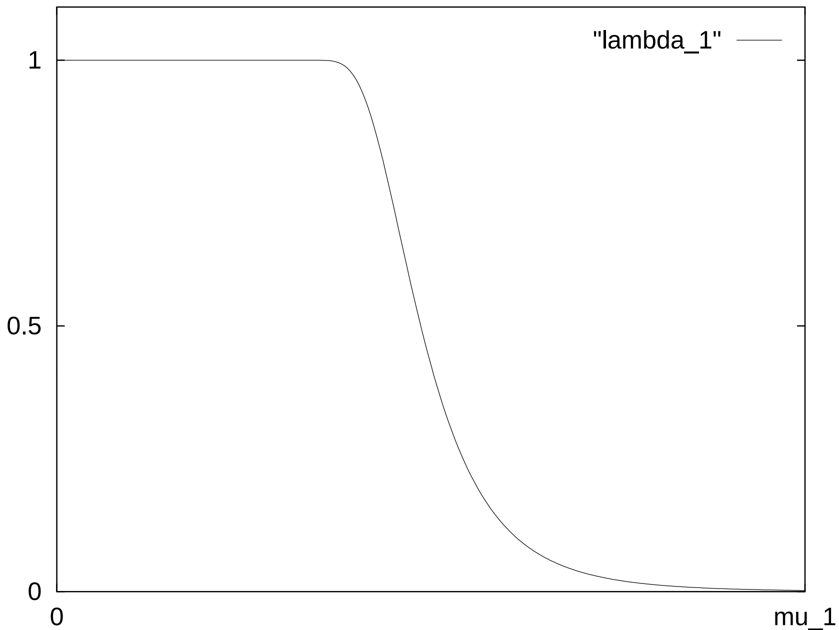
Eigenvectors: $v_1 \parallel \nabla u_\sigma, \quad v_2 \perp \nabla u_\sigma$ (normalised)

Eigenvalues: diffusion along edge: $\lambda_2 \ := \ 1$

diffusion across edge: $\lambda_1 \ := \ g(|\nabla u_\sigma|^2)$
with a diffusivity g as before.

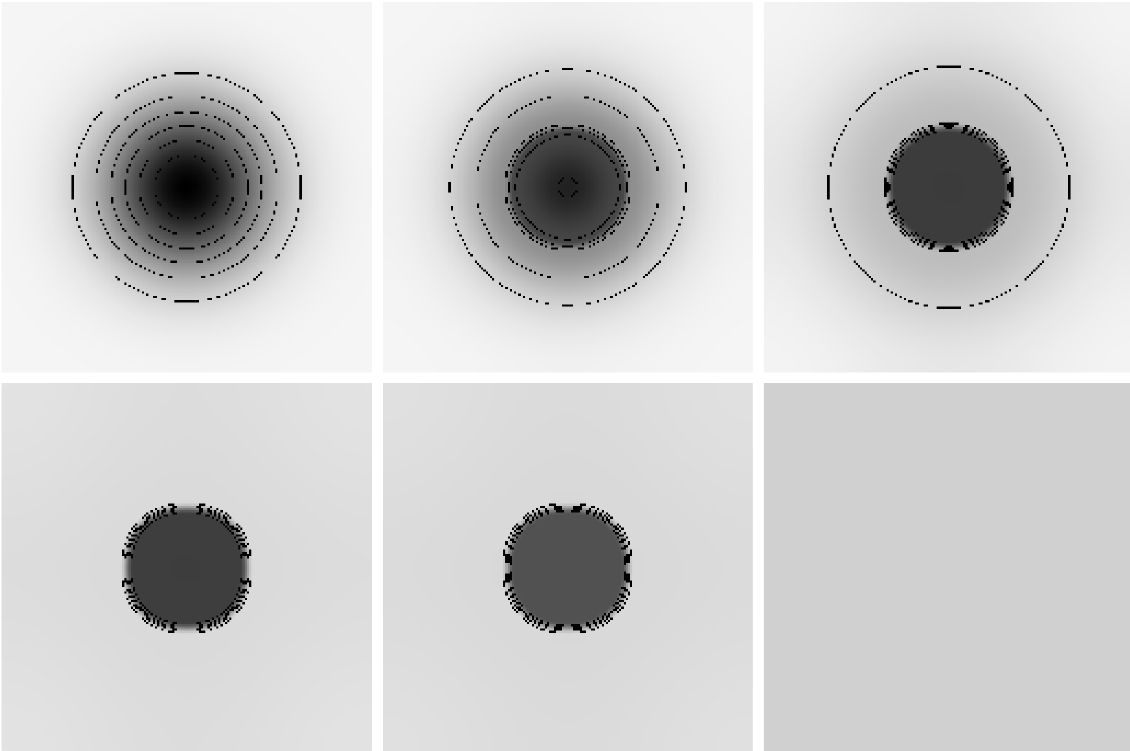
The eigenvectors v_1, v_2 and their eigenvalues λ_1, λ_2 determine the diffusion tensor:

$$D(\nabla u_\sigma) \ := \ (v_1|v_2) \, \mathrm{diag}(\lambda_1, \lambda_2) \begin{pmatrix} v_1^\top \\ v_2^\top \end{pmatrix}$$



For edge-enhancing diffusion, the eigenvalue λ_1 determines diffusion across the edge. It is a smooth, decreasing function of the image contrast $|\nabla u_\sigma|^2$.

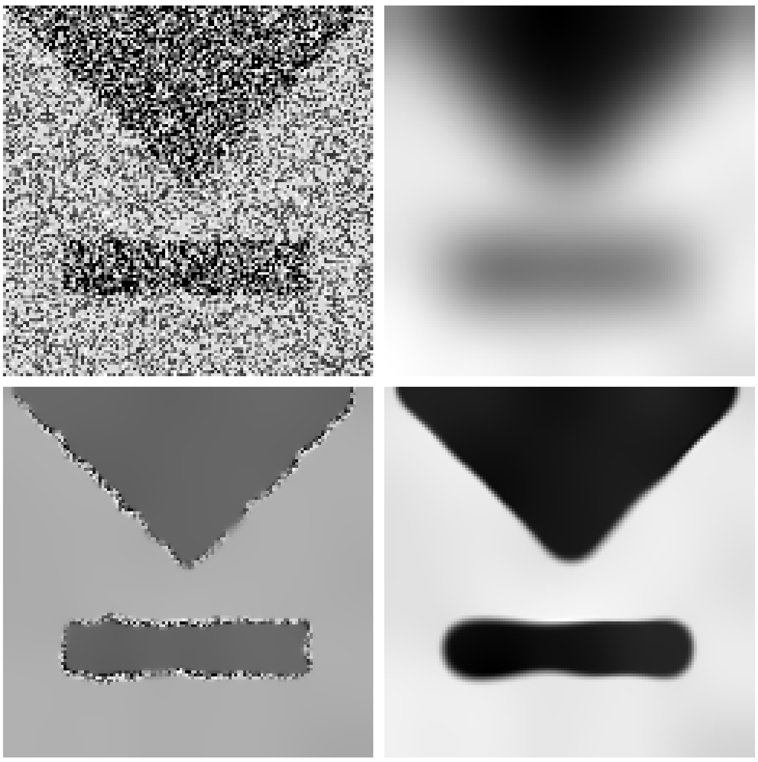
Edge-Enhancing Anisotropic Diffusion (3)



Edge-enhancing anisotropic diffusion filtering of a Gaussian-type function, $\Omega = (0, 256)^2$, $\lambda = 3.6$, $\sigma = 2$. **From top left to bottom right:** $t = 0$, 125, 625, 3125, 15625, 78125.

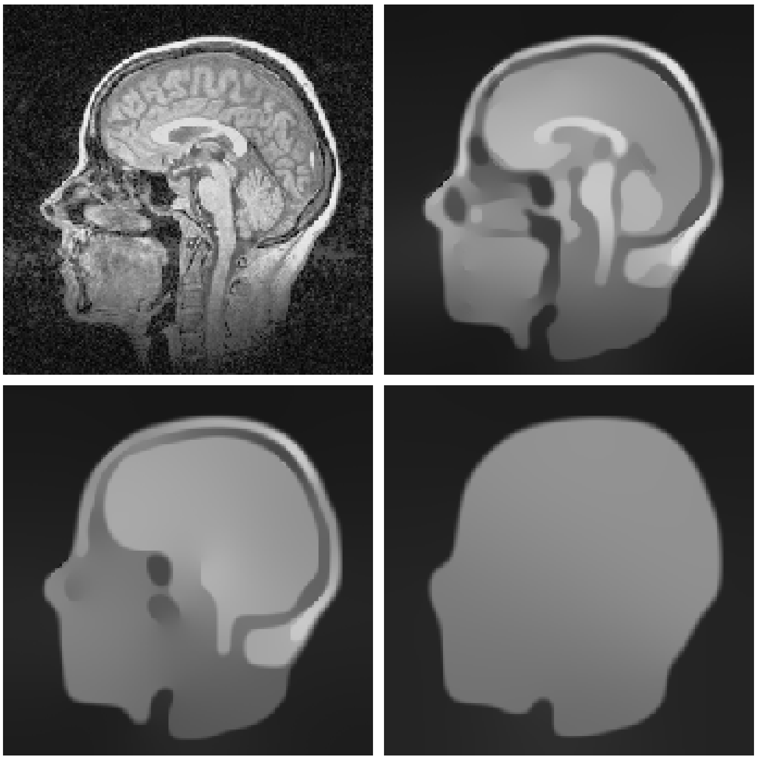
MIA 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

Edge-Enhancing Anisotropic Diffusion (4)



Restoration properties of diffusion filters. (a) **Top left:** Test image, $\Omega = (0, 128)^2$. (b) **Top right:** Linear diffusion, $t = 80$. (c) **Bottom left:** Nonlinear isotropic diffusion, $\lambda = 3.5$, $\sigma = 3$, $t = 80$. (d) **Bottom right:** Nonlinear anisotropic diffusion, $\lambda = 3.5$, $\sigma = 3$, $t = 80$.

Edge-Enhancing Anisotropic Diffusion (5)



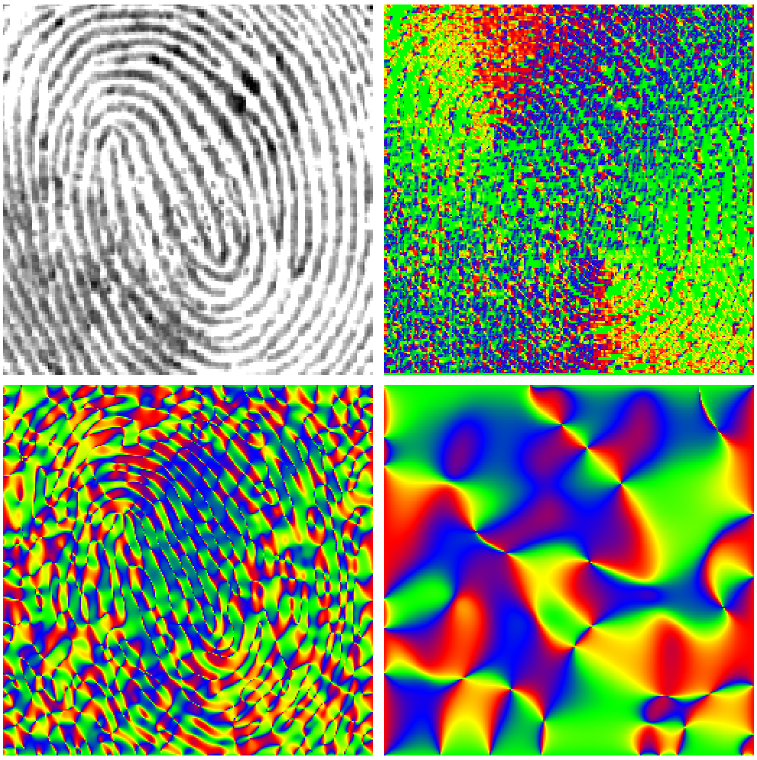
Scale-space behaviour of nonlinear anisotropic diffusion filtering ($\lambda = 3$, $\sigma = 1$). (a) **Top left:** Original image, $\Omega = (0, 236)^2$. (b) **Top right:** $t = 250$. (c) **Bottom left:** $t = 875$. (d) **Bottom right:** $t = 3000$.

Edge-Enhancing Anisotropic Diffusion (6)

Limitations

EED is not good for closing interrupted lines in flow-like structures. (a) **Left:** Original fingerprint, $\Omega = (0, 256)^2$. (b) **Right:** Edge-enhancing anisotropic diffusion, $\sigma = 1$, $\lambda = 4$, $t = 20$.

Edge-Enhancing Anisotropic Diffusion (7)



Analysis of the regularised gradient direction. (a) **Top left:** Original image, $\Omega = (0, 256)^2$. (b) **Top right:** Gradient direction, colour coded. (c) **Bottom left:** Gradient direction after Gaussian convolution with $\sigma = 4$. (d) **Bottom right:** $\sigma = 16$.

Edge-Enhancing Anisotropic Diffusion (8)

EED is not good for closing interrupted lines in flow-like structures.
It seems that the diffusion goes in inappropriate directions.

Reason

- large convolution kernels cancel adjacent gradients with the same direction but opposite orientation
- resulting smoothed flow field is of no use

$\Longrightarrow$ need more sophisticated structure descriptor: *structure tensor*

Analysing Local Structure

Smoothed image gradient $\nabla u_\sigma := \nabla K_\sigma * u$ is not optimal:

- small σ: sensitive to noise
- large σ: cancellation of adjacent gradients having the same direction, but opposite orientation.

Remedy: (Förstner/Gülch 1987)

- Consider the matrix $\nabla u_\sigma \nabla u_\sigma^\top$ instead of ∇u_σ.
  This solves the cancellation problem since $\nabla u_\sigma \nabla u_\sigma^\top = (-\nabla u_\sigma)(-\nabla u_\sigma^\top)$.
- Average the direction on some integration scale ρ:

$$J_\rho(\nabla u_\sigma) := K_\rho * (\nabla u_\sigma \nabla u_\sigma^\top)$$

  This matrix is called *structure tensor*.

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

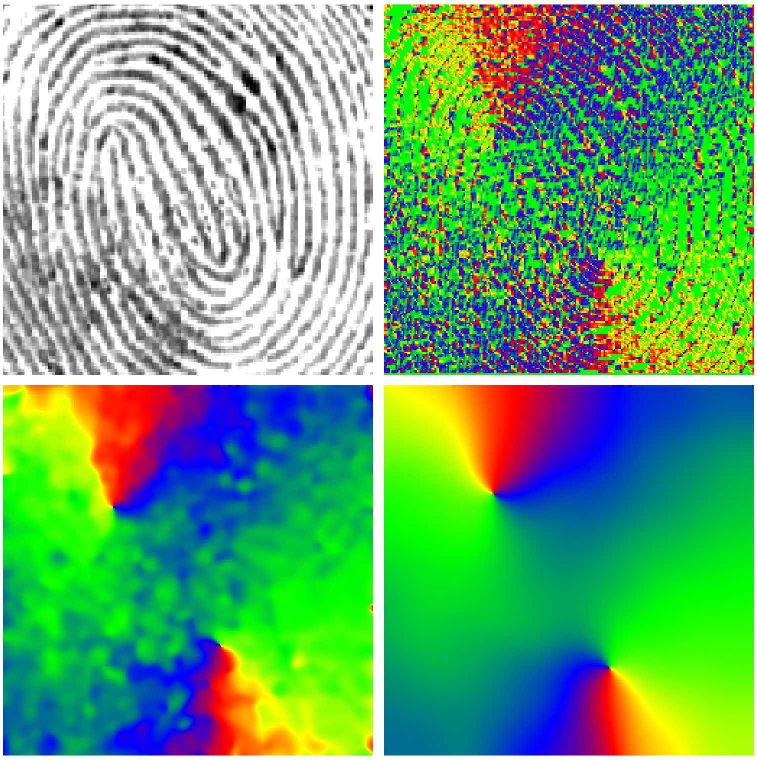
Properties of the Structure Tensor

- J_ρ is symmetric and positive semidefinite.
- eigenvectors v_1, v_2: preferred local structure directions
  eigenvalues μ_1, μ_2: local contrast along these directions (w.l.o.g.: $\mu_1 \geq \mu_2$)
- eigenvalues useful for analysing structures:

  flat areas: $\mu_1 = \mu_2 = 0$
  straight edges: $\mu_1 \gg \mu_2 = 0$
  corners: $\mu_1 \geq \mu_2 \gg 0$
  coherence (anisotropy): $(\mu_1 - \mu_2)^2$

- two smoothing scales:
  - *noise scale σ:*
    – removes noise and small-scale details
    – should be chosen very small in order to avoid cancellation effects
  - *integration scale ρ:*
    – averages directional information
    – usually much larger than σ

Analysing Local Structure (3)



Directional analysis using the structure tensor. (a) **Top left:** Original image, $\Omega = (0, 256)^2$. (b) **Top right:** Direction of the eigenvector with the smaller eigenvalue. Integration scale: $\rho = 0$. (c) **Bottom left:** $\rho = 4$. (d) **Bottom right:** $\rho = 16$.

Coherence-Enhancing Anisotropic Diffusion (1)

Coherence-Enhancing Anisotropic Diffusion (CED)

Goals:

- smooth mainly along coherent structures (i.e. along v_2)
- increase smoothing with coherence $(\mu_1 - \mu_2)^2$

Choice of the Diffusion Tensor $D(J_\rho)$

Eigenvectors: same eigenvectors as $J_\rho(\nabla u_\sigma)$: v_1, v_2

Eigenvalues: across coherent structures:
$\lambda_1 := \alpha$ (small $\alpha > 0$)

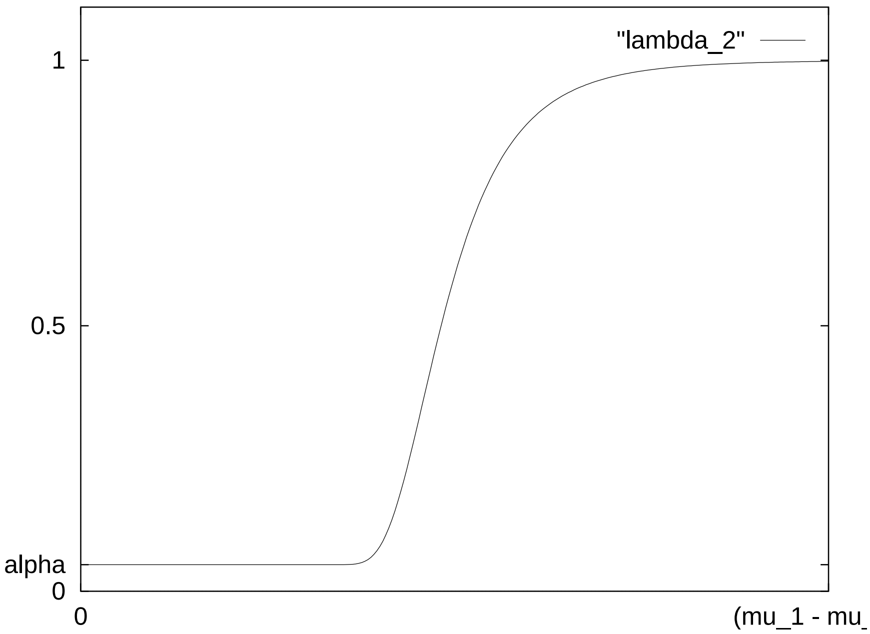
along coherent structures:

$$\lambda_2 := \begin{cases} \alpha & \text{if } \mu_1 = \mu_2, \\ \alpha + (1-\alpha)\exp\left(\frac{-C}{(\mu_1-\mu_2)^2}\right) & \text{else.} \end{cases}$$

The parameter $\alpha > 0$ ensures some small amount of isotropic diffusion that is required for proving well-posedness. The $\exp$ function is chosen for smoothness reasons.

MIA 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26

Coherence-Enhancing Anisotropic Diffusion (2)

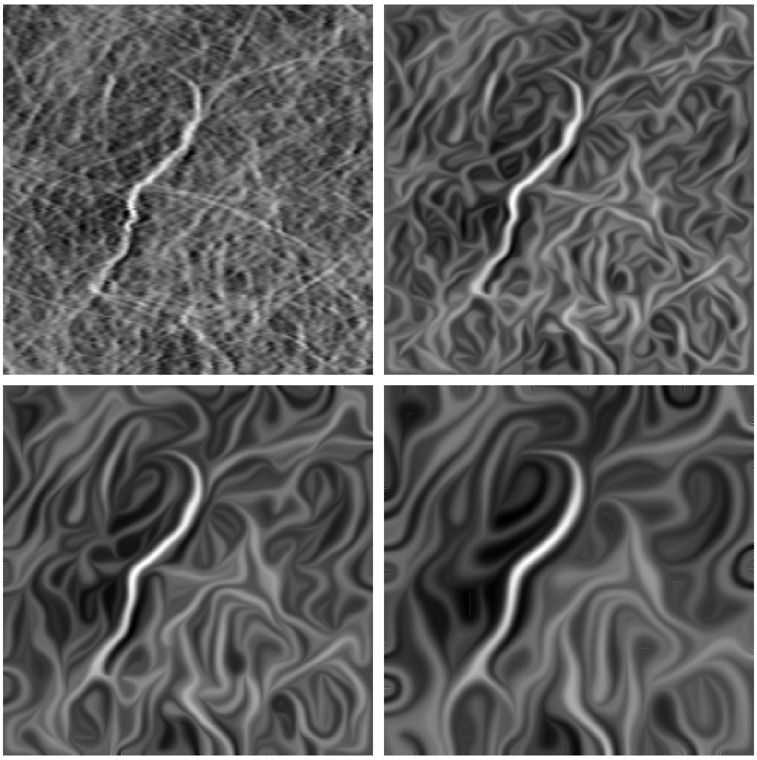


For coherence-enhancing diffusion, the eigenvalue λ_2 determines diffusion along the coherence direction v_2. It is a smooth, increasing function of the coherence measure $(\mu_1 - \mu_2)^2$.

Coherence-Enhancing Anisotropic Diffusion (3)

Fingerprint enhancement. (a) **Left:** Original fingerprint image, $\Omega = (0, 256)^2$. (b) **Right:** Filtered with coherence-enhancing anisotropic diffusion, $\sigma = 0.5$, $\rho = 4$, $t = 20$.

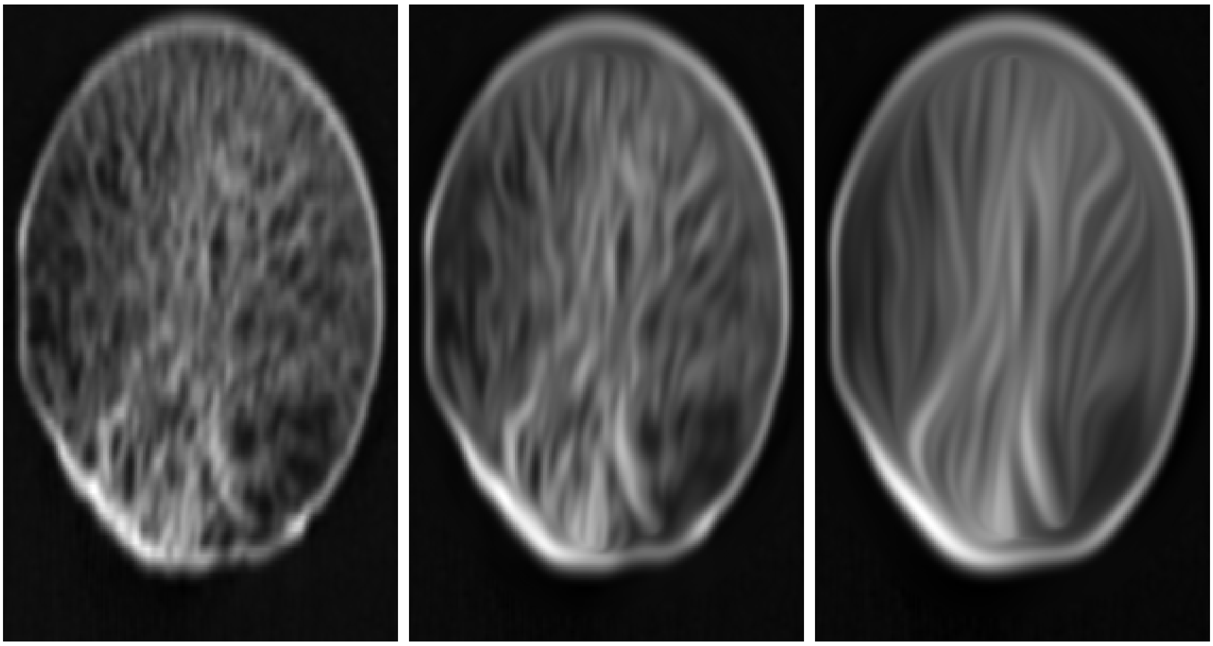
Coherence-Enhancing Anisotropic Diffusion (4)



Scale-space behaviour of coherence-enhancing diffusion ($\sigma = 0.5$, $\rho = 2$). (a) **Top left:** Original fabric image, $\Omega = (0, 257)^2$. (b) **Top right:** $t = 20$. (c) **Bottom left:** $t = 120$. (d) **Bottom right:** $t = 640$.

Coherence-Enhancing Anisotropic Diffusion (5)

M I A
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26



(a) **Left:** High resolution slipring CT scan of a femural bone, $\Omega = (0, 186) \times (0, 300)$. (b) **Middle:** Filtered by coherence-enhancing anisotropic diffusion, $\sigma = 0.5$, $\rho = 6$, $t = 16$. (c) **Right:** Ditto with $t = 128$.

Coherence-Enhancing Anisotropic Diffusion (6)

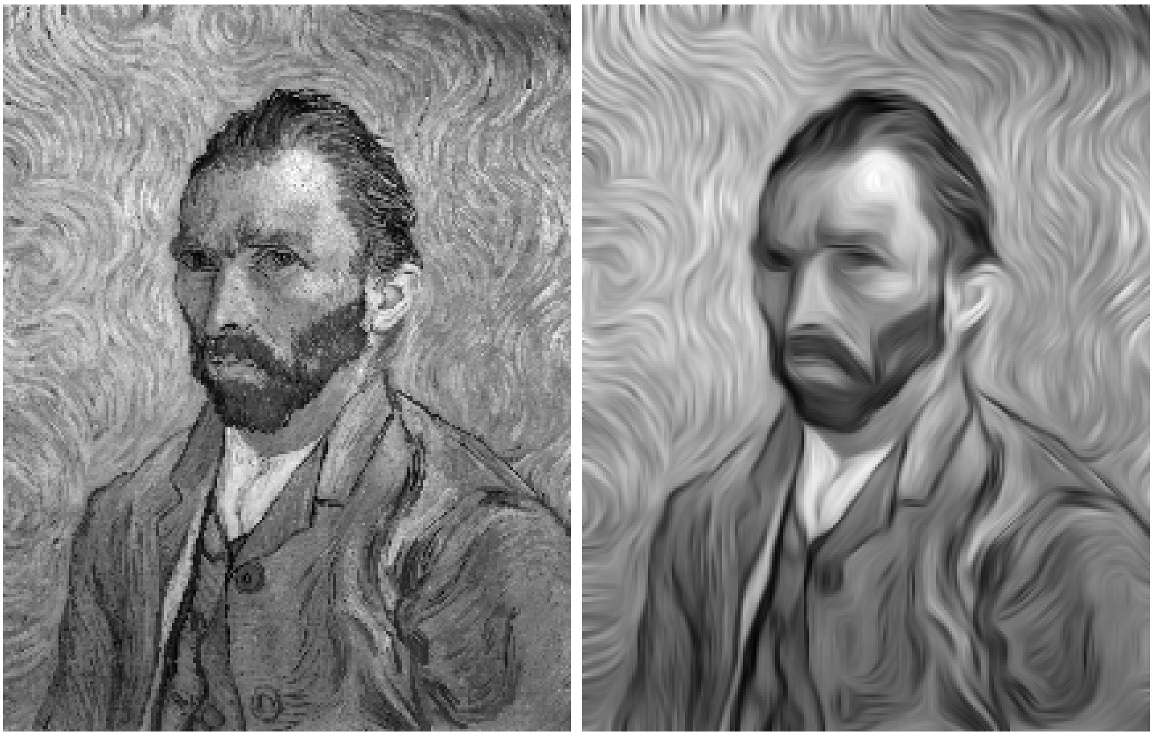
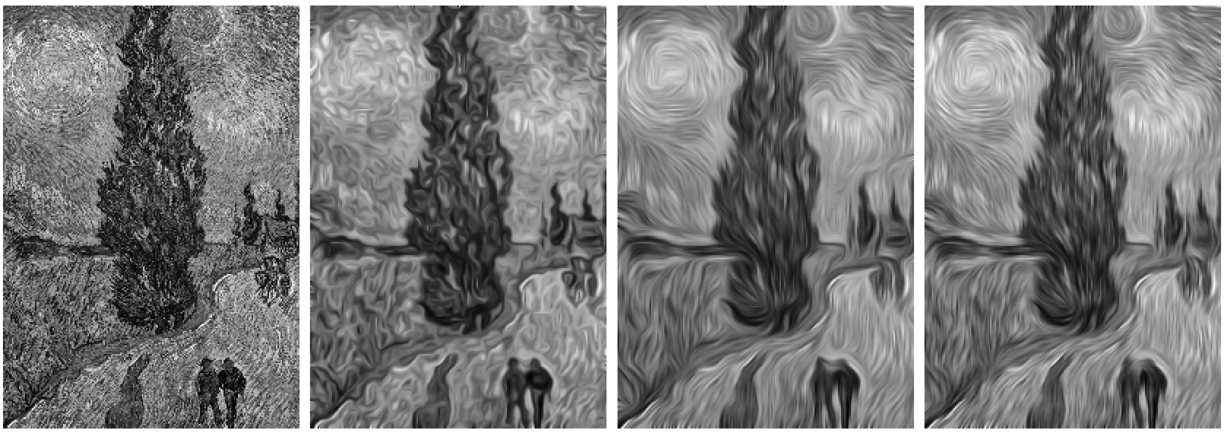


Image restoration using coherence-enhancing anisotropic diffusion. (a) **Left:** "Selfportrait" by van Gogh (Saint-Rémy, 1889; Paris, Museé d'Orsay), $\Omega = (0, 215) \times (0, 275)$. (b) **Right:** Filtered, $\sigma = 0.5$, $\rho = 4$, $t = 6$.

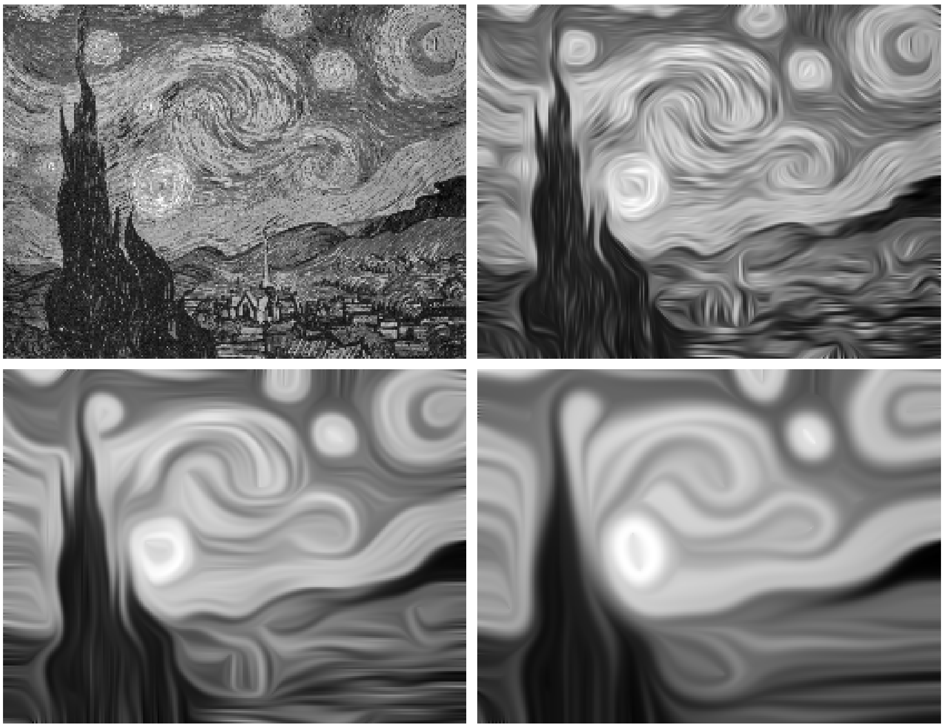
Coherence-Enhancing Anisotropic Diffusion (7)

M I A 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26



Impact of the integration scale on coherence-enhancing anisotropic diffusion ($\sigma = 0.5$, $t = 8$). (a) **Left:** "Lane under Cypresses below the Starry Sky" by van Gogh (Auvers-sur-Oise, 1890; Otterlo, Rijksmuseum Kröller–Müller), $\Omega = (0, 203) \times (0, 290)$. (b) **Middle left:** Filtered with $\rho = 1$. (c) **Middle right:** $\rho = 4$. (d) **Right:** $\rho = 6$.

Coherence-Enhancing Anisotropic Diffusion (8)



Scale-space properties of coherence-enhancing anisotropic diffusion ($\sigma = 0.5$, $\rho = 4$). (a) **Top left:** "Starry Night" by van Gogh (Saint-Rémy, 1889; New York, Museum of Modern Art), $\Omega = (0, 255) \times (0, 199)$. (b) **Top right:** $t = 8$. (c) **Bottom left:** $t = 64$. (d) **Bottom right:** $t = 512$.

Summary

Summary

- Two anisotropic models:
  - Edge-enhancing diffusion (EED):
    inhibits smoothing across edges, but not along them
  - Coherence-enhancing diffusion (CED):
    basically a 1-D smoothing along flow-like structures
- Structure tensor:
  - generalises gradient
  - eigenvalues and eigenvectors:
    for detecting corners and coherent structures
  - necessary for steering CED
- Anisotropic diffusion is applicable to
  - medical imaging
  - computer aided quality control

References

- A. R. Rao, B. G. Schunck: Computing oriented texture fields. *CVGIP: Graphical Models and Image Processing*, Vol. 53, 157–185, 1991.
  (detailed explanation of the structure tensor)
- J. Weickert: Theoretical foundations of anisotropic diffusion in image processing. *Computing*, Supplement 11, 221–236, 1996. `www.mia.uni-saarland.de/weickert/Papers/dag94.ps.gz`
  (introduced edge-enhancing anisotropic diffusion)
- J. Weickert: Coherence-enhancing diffusion filtering. *International Journal of Computer Vision*, Vol. 31, No. 2/3, 111–127, 1999. `www.mia.uni-saarland.de/weickert/Papers/ced.ps.gz`
  (details on coherence-enhancing diffusion)
- J. Weickert, *Anisotropic Diffusion in Image Processing*, Teubner, Stuttgart, 1998.
  (The structure tensor is sketched in Section 2.2, and EED and CED are investigated in Chapter 5.)
  The book is based on a Ph.D. thesis which may be downloaded from
  `www.mia.uni-saarland.de/weickert/Papers/diss.ps.gz`

Assignment C2 – Classroom Work

Problem 1 (Stencils and Discrete Diffusion Properties)

Investigate which of the following stencils satisfy the requirements (D1)–(D6) of a discrete diffusion process in the sense of Lecture 5. If some of these requirements are violated, which scale-space properties are no longer satisfied when the corresponding filter is applied iteratively?

(a)

0	$\frac{1}{8}$	0
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
0	$\frac{3}{8}$	0

(b)

$\frac{1}{7}$	$\frac{1}{7}$	0
$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
0	$\frac{1}{7}$	$\frac{1}{7}$

(c)

0	$\frac{1}{8}$	0
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
0	$\frac{1}{8}$	0

(d)

$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$\frac{1}{8}$	$\frac{1}{8}$	0

(e)

0	$\frac{1}{2}$	0
$\frac{1}{2}$	-1	$\frac{1}{2}$
0	$\frac{1}{2}$	0

(f)

0	$\frac{1}{4}$	0
$\frac{1}{4}$	0	$\frac{1}{4}$
0	$\frac{1}{4}$	0

(g)

$\frac{1}{8}$	0	$\frac{1}{8}$
0	$\frac{1}{2}$	0
$\frac{1}{8}$	0	$\frac{1}{8}$

(h)

0	$\frac{1}{4}$	0
0	$\frac{1}{2}$	0
0	$\frac{1}{4}$	0

Problem 2 (AOS Scheme)

Show that the AOS scheme differs from the semi-implicit scheme by terms that are at least of quadratic order in the time step size.