Lecture 5:

Nonlinear Isotropic Diffusion Filtering II: Semidiscrete and Discrete Theory

Contents

Motivation

- In Lecture 4, we have derived a well-posedness and scale-space theory for isotropic nonlinear diffusion filters in the *continuous* setting.
- Do similar results hold for the algorithmically relevant *discrete* setting ?
- We proceed in two steps:
  - semidiscrete case: discrete in space, continuous in time
  - fully discrete case: discrete in space and time
- This will lead to design criteria for reliable diffusion algorithms.

Semidiscrete Well-Posedness and Scale-Space Theory

Example

standard spatial discretisation of regularised diffusion equation

$$\begin{aligned}\partial_t u &= \operatorname{div}\left(g(|\nabla u_\sigma|^2)\,\nabla u\right)\\ &= \partial_x\left(g(|\nabla u_\sigma|^2)\,\partial_x u\right) + \partial_y\left(g(|\nabla u_\sigma|^2)\,\partial_y u\right)\end{aligned}$$

in some inner pixel (i,j) yields the ordinary differential equation

$$\begin{aligned}\frac{du_{i,j}}{dt} &= \frac{1}{h_1}\left(\frac{g_{i+1,j}+g_{i,j}}{2}\,\frac{u_{i+1,j}-u_{i,j}}{h_1} - \frac{g_{i,j}+g_{i-1,j}}{2}\,\frac{u_{i,j}-u_{i-1,j}}{h_1}\right)\\ &+ \frac{1}{h_2}\left(\frac{g_{i,j+1}+g_{i,j}}{2}\,\frac{u_{i,j+1}-u_{i,j}}{h_2} - \frac{g_{i,j}+g_{i,j-1}}{2}\,\frac{u_{i,j}-u_{i,j-1}}{h_2}\right).\end{aligned}$$

Since this equation is still continuous in the time t, it is called a *semidiscretisation* of the diffusion equation.

Properties of this Semidiscretisation

- even valid at the boundary pixels, if Neumann boundary conditions are realised by reflected dummy pixels (cf. Lecture 2)
- $g_{i,j}$ approximates the diffusivity $g(|\nabla u_\sigma|^2) = g(|\nabla(K_\sigma * u)|^2)$ in pixel (i,j)
- Since g is assumed to be a C^∞ function (Lecture 4), $g_{i,j}$ is a nonlinear C^∞ function of the entire set of pixel values $\{u_{k,l} \mid k=1,...,n;\ l=1,...,m\}$.
- Even in the Perona-Malik case (i.e. without Gaussian regularisation), $g_{i,j}$ remains a nonlinear C^∞ function.

Example in More Compact Notation

Representing pixel (i,j) by a single index $k(i,j)$ leads to

$$\frac{du_k}{dt} = \sum_{n=1}^{2} \sum_{l \in \mathcal{N}_n(k)} \frac{g_l + g_k}{2h_n^2}\,(u_l - u_k),$$

where $\mathcal{N}_n(k)$ are the neighbours of pixel k in n-direction
(boundary pixels may have less neighbours).

This can be written as a system of ordinary differential equations (ODEs):

$$\frac{d\mathbf{u}}{dt} = A(\mathbf{u})\,\mathbf{u},$$

where $\mathbf{u} = (u_1, ..., u_N)^\top$, and the $N \times N$ matrix $A(\mathbf{u}) = (a_{k,l}(\mathbf{u}))$ satisfies

$$a_{k,l} := \begin{cases} \frac{g_k+g_l}{2h_n^2} & (l \in \mathcal{N}_n(k)), \\ -\sum_{n=1}^{2} \sum_{l \in \mathcal{N}_n(k)} \frac{g_k+g_l}{2h_n^2} & (l = k), \\ 0 & (\text{else}). \end{cases}$$

Note that $a_{k,l}$ is C^∞ for all k and l, since g is C^∞.

A Semidiscrete Theoretical Framework

We would like to obtain a similar well-posedness and scale-space theory as in the continuous case (Lecture 4).

Assumptions

Let $\mathbf{f} \in \mathbb{R}^N$ and consider the ODE system

$$\begin{aligned} \frac{d\mathbf{u}}{dt} &= A(\mathbf{u})\,\mathbf{u}, \\ \mathbf{u}(0) &= \mathbf{f}. \end{aligned}$$

Semidiscrete Well-Posedness and Scale-Space Theory (5)

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$A = (a_{i,j})$ is assumed to have the following properties:

- **(S1) Smoothness:**
  A is a Lipschitz-continuous function of $\mathbf{u}$, for every bounded subset of $\mathbb{R}^N$.
  (i.e. there exists some L with $|A\mathbf{u} - A\mathbf{v}| \le L|\mathbf{u} - \mathbf{v}|$ for all $\mathbf{u}$, $\mathbf{v}$ within this subset; e.g. satisfied if $a_{i,j} : \mathbb{R}^N \to \mathbb{R}$ is a C^1-function for all i, j.)

- **(S2) Symmetry:** $a_{i,j}(\mathbf{u}) = a_{j,i}(\mathbf{u})$

- **(S3) Vanishing Row Sums:** $\sum_j a_{i,j}(\mathbf{u}) = 0 \quad \forall\, i$

- **(S4) Nonnegative Off-Diagonals:** $a_{i,j}(\mathbf{u}) \ge 0 \quad \forall\, i \ne j$

- **(S5) Irreducibility:**
  For any i, j there exist $k_0, \ldots, k_r$ with $k_0 = i$ and $k_r = j$ such that $a_{k_p, k_{p+1}} \ne 0$ for $p = 0, \ldots, r-1$. ("There is some way from pixel i to pixel j")

Semidiscrete Well-Posedness and Scale-Space Theory (6)

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Results

(a) **Well-Posedness**
Existence of a unique solution which depends continuously on $\mathbf{f}$ and the right hand side of the ODE system

(b) **Average Grey Level Invariance**

$$\frac{1}{N}\sum_{j=1}^{N} u_j(t) \;=\; \frac{1}{N}\sum_{j=1}^{N} f_j \;=:\; \mu \qquad \forall\, t > 0.$$

(c) **Maximum–Minimum Principle**

$$\min_j f_j \;\le\; u_i(t) \;\le\; \max_j f_j \qquad \forall\, i,\ \forall\, t > 0.$$

(d) **Lyapunov Functions**

$$V(t) \;:=\; \sum_i r(u_i(t))$$

is a Lyapunov function for all convex $r \in \mathrm{C}^1$: decreasing and bounded from below.

(e) **Convergence to a Constant Steady-State**

$$\lim_{t\to\infty} u_i(t) = \mu \qquad \forall\, i$$

Application to the Example

The standard spatial finite difference (FD) discretisation of a continuous isotropic diffusion filter satisfying (C1'),(C3') fulfils the semidiscrete requirements (S1)–(S5):

- Smoothness (S1) follows from the fact that $a_{k,l}$ is a C^∞ function of $\mathbf{u}$ (cf. Page 5)
- The symmetry (S2) is easy to check: For $k \neq l$ we have

$$\begin{aligned} a_{k,l} \;&=\; \frac{g_k+g_l}{2h_n^2} \;=\; a_{l,k} \qquad && (l \in \mathcal{N}_n(k)), \\ a_{k,l} \;&=\; 0 \;=\; a_{l,k} \qquad && (l \notin \mathcal{N}_n(k)). \end{aligned}$$

- Vanishing row sums (S3) follow from

$$a_{k,k} \;=\; -\sum_{n=1}^{2} \sum_{l \in \mathcal{N}_n(k)} \frac{g_k+g_l}{2h_n^2} \;=\; -\sum_{l, l\neq k} a_{k,l} \qquad \forall\, k.$$

- Nonnegative off-diagonals (S4) are guaranteed by the definition of $a_{k,l}$ on Page 5.
- Irreducibility (S5) is satisfied: For two pixels i and j, consider any connecting path along neighbours. For two such neighbours k and l we have $a_{k,l} = \frac{g_k+g_l}{2h_n^2} > 0$.

Surprising Consequence:

A standard spatial discretisation of the unregularised Perona–Malik filter – which suffers from some well-posedness problems in the continuous setting – does also satisfy (S1)–(S5). Hence:

A spatially discretised Perona–Malik filter is well-posed and creates a scale-space transformation.

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Discrete Theory

Examples

Let the nonlinear ODE system $\frac{d\mathbf{u}}{dt} = A(\mathbf{u})\mathbf{u}$ satisfy (S1)–(S5).
We want to study two different time discretisations.

1. **Explicit Scheme**

$$\frac{\mathbf{u}^{k+1} - \mathbf{u}^k}{\tau} = A(\mathbf{u}^k)\,\mathbf{u}^k$$

leads to matrix-vector multiplication (A is a nonlinear C^∞ function of $\mathbf{u}^k$):

$$\mathbf{u}^{k+1} = \left(I + \tau A(\mathbf{u}^k)\right)\,\mathbf{u}^k.$$

Matrix elements depend nonlinearly on $\mathbf{u}$.

2. **Semi-Implicit Scheme**

$$\frac{\mathbf{u}^{k+1}-\mathbf{u}^{k}}{\tau} \;=\; A(\mathbf{u}^{k})\,\mathbf{u}^{k+1}$$

requires to solve a linear system of equations:

$$\big(I-\tau A(\mathbf{u}^{k})\big)\,\mathbf{u}^{k+1} \;=\; \mathbf{u}^{k}.$$

Can also be regarded as matrix-vector multiplication:

$$\mathbf{u}^{k+1} \;=\; \big(I-\tau A(\mathbf{u}^{k})\big)^{-1}\,\mathbf{u}^{k}.$$

A Discrete Theoretical Framework

Assumptions

Let $\mathbf{f}\in\mathbb{R}^N$ and compute $(\mathbf{u}^k)_{k\in\mathbb{N}_0}$ by

$$\begin{aligned}\mathbf{u}^0 &= \mathbf{f},\\ \mathbf{u}^{k+1} &= Q(\mathbf{u}^k)\,\mathbf{u}^k, \qquad \forall\, k\in\mathbb{N}_0,\end{aligned}$$

where the $N\times N$ matrix $Q=(q_{i,j})$ is a (nonlinear) function of $\mathbf{u}^k$ that satisfies

- **(D1) Smoothness:** $q_{i,j}(\mathbf{u}^k)$ is continuous in $\mathbf{u}^k$.
- **(D2) Symmetry:** $q_{i,j}(\mathbf{u}^k) = q_{j,i}(\mathbf{u}^k)$
- **(D3) Unit Row Sums:** $\sum_j q_{i,j}(\mathbf{u}^k) = 1 \quad \forall i$
- **(D4) Nonnegativity:** $q_{i,j}(\mathbf{u}^k) \geq 0$
- **(D5) Positive Diagonal:** $q_{i,i}(\mathbf{u}^k) > 0$
- **(D6) Irreducibility**

Results

(a) **Well-Posedness**
Unique solution $\mathbf{u}^k$ depends continuously on $\mathbf{f}$, for every $k > 0$.

(b) **Average Grey Level Invariance**

$$\frac{1}{N}\sum_{j=1}^{N} u_j^k \;=\; \frac{1}{N}\sum_{j=1}^{N} f_j \;=:\; \mu \qquad \forall\, k \in \mathbb{N}_0.$$

(c) **Maximum–Minimum Principle**

$$\min_j f_j \;\le\; u_i^k \;\le\; \max_j f_j \qquad \forall\, i,\; \forall\, k > 0.$$

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

(d) **Lyapunov Sequences**

$$V^k \;:=\; \sum_i r(u_i^k)$$

is a Lyapunov sequence for all convex $r \in \mathrm{C}$:
V^k is decreasing in k and bounded from below.

(e) **Convergence to a Constant Steady-State**

$$\lim_{k\to\infty} u_i^k = \mu \qquad \forall\, i.$$

Application to the Examples

The discrete requirements (D1)–(D6) constitute simple design criteria for reliable nonlinear diffusion algorithms.

Let $A = (a_{i,j}) : \mathbb{R}^N \to \mathbb{R}^{N \times N}$ satisfy the semidiscrete requirements (S1)–(S5).
Let us now check if explicit or semi-implicit discretisations satisfy (D1)–(D6).

1. **Explicit Scheme**

$$\mathbf{u}^{k+1} \;=\; \underbrace{\left(I + \tau A(\mathbf{u}^k)\right)}_{Q(\mathbf{u}^k)} \mathbf{u}^k$$

fulfils immediately the discrete requirements (D1)–(D3) and (D6):

- Smoothness of Q (D1) follows from the smoothness of A (S1).
- Symmetry of Q (D2) follows from the symmetry of A (S2).
- Unit row sums for Q (D3) follow from vanishing row sums for A (S3).
- Irreducibility for Q (D6) follows from irreducibility for A (S5):
  Note that $q_{k,l} > 0$ if $a_{k,l} > 0$.

The nonnegativity / positivity requirements (D4) and (D5) are satisfied if

$$q_{i,i} \;=\; 1 + \tau a_{i,i} \;>\; 0 \qquad \forall\, i.$$

This yields (note that $a_{i,i} < 0$)

$$\tau \;<\; \frac{1}{\max_i |a_{i,i}(\mathbf{u}^k)|}.$$

Using

$$a_{i,i} \;=\; -\sum_{n=1}^{2} \sum_{k \in \mathcal{N}_n(i)} \frac{g_k + g_i}{2h_n^2}$$

and $h_n := 1$ as well as $0 < g_j \le 1$, we get

$$|a_{i,i}| \;\le\; \sum_{n=1}^{2} \sum_{k \in \mathcal{N}_n(i)} 1 \;=\; 4.$$

Thus, the explicit scheme creates a discrete scale-space for

$$\tau \;<\; \frac{1}{4}.$$

Same severe restriction as in the linear diffusion case (Lecture 2):
Many time steps are necessary for reaching some "interesting" diffusion time.

2. **Semi-Implicit Scheme**

$$\mathbf{u}^{k+1} = \left(I - \tau A(\mathbf{u}^k)\right)^{-1} \mathbf{u}^k$$

One can show that (D1)–(D6) are satisfied for arbitrarily (!) large time steps τ. It can be an interesting alternative to the explicit scheme provided that it is possible to solve the resulting linear system of equations in an efficient manner.

Summary

- well-posedness and scale-space theory for semidiscrete and fully discrete case in complete analogy to continuous case:
  - existence, uniqueness, stability w.r.t. initial image
  - extremum principle
  - average grey level invariance
  - many simplifying Lyapunov functions/sequences
  - convergence to constant steady-state
- semidiscrete framework satisfied by standard FD discretization (even for the unregularised Perona-Malik equation)
- discrete framework satisfied
  - by explicit scheme for times step sizes $\tau < 0.25$
  - by semi-implicit scheme for all time step sizes

References

- J. Weickert, *Anisotropic Diffusion in Image Processing*, Teubner, Stuttgart, 1998. See also `www.mia.uni-saarland.de/weickert/Papers/diss.ps.gz`
(Chapters 3 and 4 present a slightly more general semidiscrete and discrete theory covering also anisotropic diffusion filters.)