Lecture 3:

Linear Diffusion Filtering II: Numerical Aspects, Limitations, Alternatives

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Numerical Aspects

images are sampled on a pixel grid
$\Longrightarrow$ discrete diffusion filtering is required

- *efficient for large kernel sizes σ:*
  – Fast Fourier Transform (FFT)
  – separable
  – computational effort independent of σ: $O(N \log N)$ for image with N pixels
- *typical solution in the spatial domain:*
  – sample Gaussian and truncate at a multiple of σ
  – separability and symmetry reduces effort to $O(N\sigma)$
  – preferable for small σ
- *discretisations of the diffusion equation:*
  – theoretically sound, but less efficient for large times
  – instructive: ideas can also be applied to other (in particular nonlinear) PDEs

Thus, let us now focus on the third possibility.

Numerical Aspects (2)

How Can One Approximate Derivatives by Finite Differences?

1-D Example. *The second derivative u'' is to be approximated in pixel i using the grey values u_{i-1}, u_i, u_{i+1} in the pixels $i-1$, i und $i+1$. Let the spatial grid size be h. How are the weights chosen?*

Taylor expansion around pixel i:

$$\begin{aligned} u_{i-1} &= u_i - hu_i' + \frac{h^2}{2}u_i'' - \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i'''' - \frac{h^5}{120}u_i''''' + O(h^6) \\ u_i &= u_i \\ u_{i+1} &= u_i + hu_i' + \frac{h^2}{2}u_i'' + \frac{h^3}{6}u_i''' + \frac{h^4}{24}u_i'''' + \frac{h^5}{120}u_i''''' + O(h^6). \end{aligned}$$

Comparison of the coefficients:

$$\begin{aligned} 0 \cdot u_i + 0 \cdot u_i' + 1 \cdot u_i'' &\overset{!}{=} \alpha_{-1}u_{i-1} + \alpha_0 u_i + \alpha_1 u_{i+1} \\ &= (\alpha_{-1} + \alpha_0 + \alpha_1) \cdot u_i \\ &+ h(-\alpha_{-1} + \alpha_1) \cdot u_i' \\ &+ \frac{1}{2}h^2(\alpha_{-1} + \alpha_1) \cdot u_i'' + O(h^3) \end{aligned}$$

Numerical Aspects (3)

leads to a linear system of equations:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2/h^2 \end{pmatrix}$$

Its solution gives the weights

$$\alpha_{-1} = \frac{1}{h^2}, \qquad \alpha_0 = -\frac{2}{h^2}, \qquad \alpha_1 = \frac{1}{h^2}.$$

Plugging this into the Taylor expansion gives

$$\frac{1}{h^2}u_{i-1} - \frac{2}{h^2}u_i + \frac{1}{h^2}u_{i+1} = u_i'' + \frac{h^2}{12}u_i'''' + O(h^4).$$

Since the leading error term $\frac{h^2}{12}u_i''''$ is quadratic in the grid size h, this is called an *approximation of consistency order* 2. (We assume that all derivatives exist and are bounded.) In order to approximate the desired term for $h \to 0$, approximations have to be of consistency order ≥ 1. Then they are called *consistent*.

Finite Difference Approximation to the 2-D Linear Diffusion Equation

linear diffusion equation:

$$\partial_t u \;=\; \partial_{xx} u + \partial_{yy} u$$

grid with size h_1, h_2, time step size τ (in image processing: often $h_1 = h_2 = 1$):

$$\begin{aligned} x_i &:= (i - \tfrac{1}{2})\; h_1 \\ y_j &:= (j - \tfrac{1}{2})\; h_2 \\ t_k &:= k\tau \\ u_{ij}^k &: \text{ approximates } u(x_i, y_j, t_k) \end{aligned}$$

finite difference approximations in (x_i, y_i, t_k):

$$\begin{aligned} \partial_t u &= \frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} \;+\; O(\tau) \\ \partial_{xx} u &= \frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{h_1^2} \;+\; O(h_1^2) \end{aligned}$$

This leads to the scheme

$$\frac{u_{i,j}^{k+1} - u_{i,j}^k}{\tau} = \frac{u_{i+1,j}^k - 2u_{i,j}^k + u_{i-1,j}^k}{h_1^2} + \frac{u_{i,j+1}^k - 2u_{i,j}^k + u_{i,j-1}^k}{h_2^2}.$$

Its consistency order in the point (x_i, y_i, t_k) is $O(\tau + h_1^2 + h_2^2)$.

The unknown $u_{i,j}^{k+1}$ at level $k+1$ can be computed *explicitly* (i.e. without solving a system of equations) from five known values at level k:

$$u_{i,j}^{k+1} = \left(1 - 2\,\frac{\tau}{h_1^2} - 2\,\frac{\tau}{h_2^2}\right) u_{i,j}^k + \frac{\tau}{h_1^2}\, u_{i+1,j}^k + \frac{\tau}{h_1^2}\, u_{i-1,j}^k + \frac{\tau}{h_2^2}\, u_{i,j+1}^k + \frac{\tau}{h_2^2}\, u_{i,j-1}^k.$$

It is therefore called *explicit finite difference scheme*.

Numerical Aspects (6)

Weights in Stencil Notation

0	$\frac{\tau}{h_2^2}$	0
$\frac{\tau}{h_1^2}$	$1 - 2\frac{\tau}{h_1^2} - 2\frac{\tau}{h_2^2}$	$\frac{\tau}{h_1^2}$
0	$\frac{\tau}{h_2^2}$	0

where the locations refer to the indices

$(i-1, j+1)$	$(i, j+1)$	$(i+1, j+1)$
$(i-1, j)$	(i, j)	$(i+1, j)$
$(i-1, j-1)$	$(i, j-1)$	$(i+1, j-1)$

Numerical Aspects (7)

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Is This Scheme Stable ?

- stencil weights sum up to 1
- If all weights are nonnegative we obtain a stable convex combination satisfying

$$\min_{n,m} u_{n,m}^k \;\le\; u_{i,j}^{k+1} \;\le\; \max_{n,m} u_{n,m}^k \qquad \forall\, k \ge 0, \quad \forall\, i,j.$$

  With $u_{n,m}^0 := f_{n,m}$ this implies the *discrete maximum–minimum principle*

$$\min_{n,m} f_{n,m} \;\le\; u_{i,j}^k \;\le\; \max_{n,m} f_{n,m} \qquad \forall\, k \ge 0, \quad \forall\, i,j.$$

- All noncentral weights are nonnegative. Requiring for the central weight that $1 - 2\frac{\tau}{h_1^2} - 2\frac{\tau}{h_2^2} \ge 0$ leads to a *stability condition* on the time step size τ:

$$\tau \;\le\; \frac{1}{\frac{2}{h_1^2} + \frac{2}{h_2^2}}$$

- For $h_1 = h_2 = 1$ one gets $\tau \;\le\; 1/4$. Thus, large diffusion times require many iterations. This scheme is not highly efficient, but simple.

Numerical Aspects (8)

What Happens at Image Boundaries ?

In practice the image domain is a rectangle Ω with boundary $\partial\Omega$.
One has to impose *boundary conditions*, e.g.

- *Dirichlet boundary conditions:*

$$u(x,y,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty).$$

- *Homogeneous Neumann boundary conditions (reflecting boundary conditions):*

$$\frac{\partial u}{\partial n}(x,y,t) = 0 \quad \text{on} \quad \partial\Omega \times [0,\infty)$$

  where n is the (outer) normal vector on $\partial\Omega$ and $\frac{\partial u}{\partial n} := n^\top \nabla u$ is the directional derivative.

Usually Neumann boundary conditions are preferred.

Numerical Aspects (9)

- For simplicity, let us focus on the 1-D Neumann case:

$$\frac{\partial u}{\partial x}(x,t) = 0 \qquad \text{for } x = 0 \text{ or } x = Nh, \text{ and } t \geq 0.$$

- Let u_i^k denote the discretisation of $u(x,t)$ in $x_i := (i - \frac{1}{2})\,h$ at time $t_k := k\tau$.
- Then the boundary conditions in $x = 0$ and $x = Nh$ are discretised as

$$\frac{u_1^k - u_0^k}{h} = 0, \qquad \frac{u_{N+1}^k - u_N^k}{h} = 0.$$

- This comes down to introducing *dummy values* u_0^k and u_{N+1}^k by mirroring:

$$u_0^k := u_1^k, \qquad u_{N+1}^k := u_N^k.$$

  Thus, for symmetry reasons there is no flux across the boundaries.
- With this boundary extension by mirroring, the explicit finite difference stencil can be used for all pixels $i = 1,...,N$.
- In 2-D one proceeds in the same way in y-direction.

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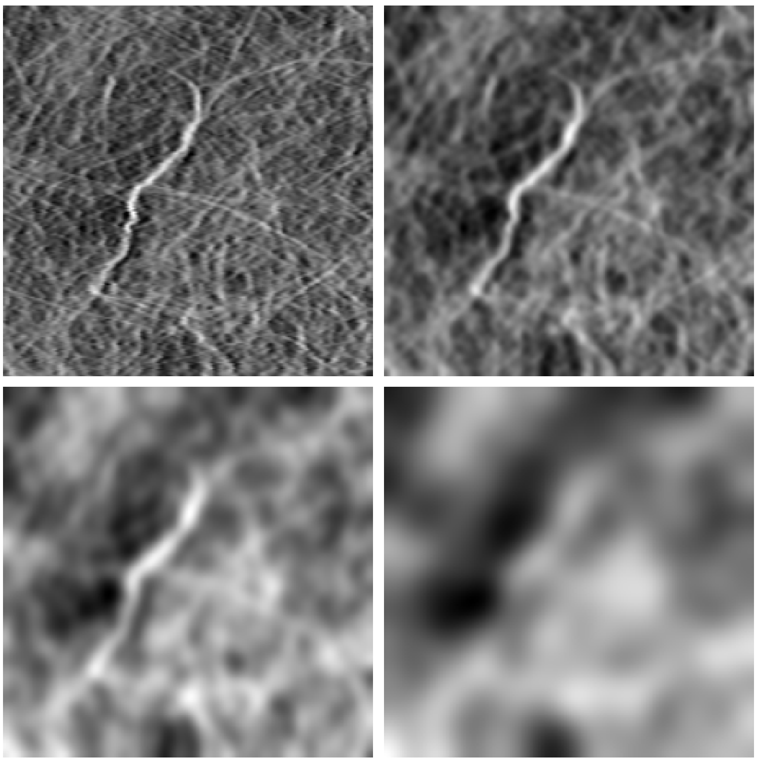
Advantages and Limitations of Gaussian Smoothing

Advantages

- strong regularising properties that are useful for stable feature detection using Gaussian derivatives (Lecture 2)
- provides a well-understood scale-space evolution (Lecture 2)
- useful for applications where images are to be simplified and edges are unimportant

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20



Application of Gaussian scale-space for analysing the cloudiness of fabrics. (a) **Top left:** Fabric image, $\Omega = (0, 257)^2$. (b) **Top right:** $t = 4$. (c) **Bottom left:** $t = 16$. (d) **Bottom right:** $t = 64$.

Advantages and Limitations of Gaussian Smoothing (3)

Shortcomings of Linear Diffusion Filtering

(a) blurs also semantically important structures such as edges

(b) structures may become dislocated

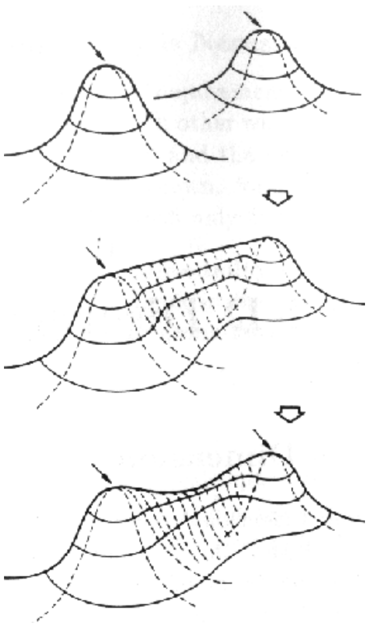
(c) can create new extrema for dimensions ≥ 2

Remedies:

(a),(b): nonlinear diffusion scale-spaces

(b),(c): morphological scale-spaces

Advantages and Limitations of Gaussian Smoothing (4)



Creation of new extrema by diffusion filtering for dimensions ≥ 2. (a) **Top:** 3D plot of an image with two maxima. The left maximum is smaller than the right one. (b) **Middle:** Connecting these extrema via a small ridge leads to an image with only one maximum. (c) **Bottom:** Diffusing the image from (b) gives an image with two maxima. **Author:** T. Lindeberg, *Discrete Scale-Space Theory and the Scale-Space Primal Sketch*, Ph.D. thesis, KTH Stockholm, Sweden, 1991.

Other Linear Scale-Spaces

How Unique is Gaussian Scale-Space ?

- Many (≥ 14) papers have been written in which Gaussian scale-space has been derived axiomatically as the only reasonable way to define a *linear* scale-space, i.e.

$$T_t(\alpha f + \beta g) \;=\; \alpha\, T_t f + \beta\, T_t g.$$

for images f, g, and scalars α, β.

- However, most of these axiomatics contain either some small error or they do not admit a sufficiently large class of scale-space operators.
- Recently, many linear alternatives to Gaussian scale-space have been advocated:
  - Poisson scale-space (Felsberg/Sommer 2001)
  - α scale-spaces (Duits et al. 2002)
  - fractional high order scale-spaces (Didas et al. 2005)
  - relativistic scale-spaces (Burgeth et al. 2005)
  - Bessel scale-space (Burgeth et al. 2005)

Poisson Scale-Space

- Another linear scale-space that satisfies all reasonable axioms of Gaussian scale-space (apart from separability) is given by the *Poisson scale-space*

$$\partial_t u \;=\; -\sqrt{-\Delta}\, u.$$

The operator $\sqrt{-\Delta}$ can be understood in the Fourier domain, where differentiation becomes multiplication with $i\omega$.

- This pseudodifferential equation corresponds to the convolution kernel

$$H_t(\mathbf{x}) \;=\; \frac{\Gamma(3/2)}{\pi^{3/2}}\, \frac{t}{(t^2 + |\mathbf{x}|^2)^{3/2}}$$

with $\Gamma(3/2) \approx 0.88623$ (Gamma function interpolating the factorial).

- discovered independently by Felsberg / Sommer (2001) and Duits *et al.* (2004).
- results are visually similar to Gaussian scale-space

Summary

- Linear diffusion filtering can be realised in a number of different ways including
  - computing Gaussian convolution as multiplication in the Fourier domain
  - convolution with a sampled Gaussian in the spatial domain
  - finite difference approximations of the diffusion equation
- The explicit finite difference scheme performs local averaging on a 3×3 stencil.
- It is applied iteratively with a time step size τ. On a 2-D unit grid it is stable (satisfies a discrete maximum–minimum principle) for $\tau \leq 0.25$.
- There are alternative linear scale-spaces that use convolution kernels, e.g. the Poisson scale-space.
- Linear diffusion filtering may blur and dislocate edges, and create new extrema for dimensions ≥ 2.

References

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  (introduced Poisson scale-space)
- R. Duits, L. Florack, J. de Graaf, B. ter Haar Romeny: On the axioms of scale space theory. *Journal of Mathematical Imaging and Vision*, Vol. 20, No. 3, 267–298, May 2004.
  (another paper on Poisson scale-space)

Assignment C1 – Classroom Work

Problem 1 (Partial Derivatives)

(a) Calculate the gradient ∇, the Hessian H, and the Laplacian Δ of the function

$$f(x, y) \;:=\; \exp(-2x^2 + 5xy - 9y^2)\;.$$

(b) Calculate the divergence div of the vector-valued function

$$\mathbf{g}(x, y) \;:=\; \begin{pmatrix} \ln(1 + x^2) \\ \sqrt{1 + x^2 + y^2} \end{pmatrix}\;.$$

Problem 2 (Convolution and Fourier Transform)

Consider

$$f(x) \;=\; \begin{cases} \frac{1}{2}, & -1 \le x \le 1 \\ 0, & \text{else}\;. \end{cases}$$

(a) Calculate $(f * f)(x)$ and $(f * f * f)(x)$ explicitly.

(b) Using $(f * f)(x)$ as found in part (a), verify the convolution theorem for the continuous Fourier transform.

Problem 3 (Otsu's Axiomatic Derivation of Gaussian Scale-Space)

Show that a transformation $\tilde{f}$ of a 2-D image $f : \mathbb{R}^2 \to \mathbb{R}$ that satisfies the 4 axioms

(1) *Representation as a linear integral operator:*
There exists a function $W : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ such that

$$\tilde{f}(\mathbf{r}) \;=\; \int_{\mathbb{R}^2} W(\mathbf{r}, \mathbf{r}')\, f(\mathbf{r}')\, d\mathbf{r}' \qquad \forall\, \mathbf{r} \in \mathbb{R}^2.$$

(2) *Translation invariance:*
For all $\mathbf{r}, \mathbf{a} \in \mathbb{R}^2$ it is required that

$$\tilde{f}(\mathbf{r}-\mathbf{a}) \;=\; \int_{\mathbb{R}^2} W(\mathbf{r}, \mathbf{r}')\, f(\mathbf{r}'-\mathbf{a})\, d\mathbf{r}'.$$

(3) *Rotation invariance (of the kernel):*
For all rotation matrices T_Θ and for all $\mathbf{r} \in \mathbb{R}^2$ it holds that $W(T_\Theta \mathbf{r}) \;=\; W(\mathbf{r})$.

(4) *Separability:*
There exists a function $u : \mathbb{R} \to \mathbb{R}$ such that $W(\mathbf{r}) \;=\; u(x)\, u(y)$ for all $\mathbf{r} = (x, y)^\top \in \mathbb{R}^2$.

is given by a convolution of f with a Gaussian-type kernel

$$W(r) \;=\; k\, \exp\left[c\,(x^2 + y^2)\right].$$