Lecture 2:
Linear Diffusion Filtering I: Basic Concepts

Contents

Physical Background of Diffusion Processes

Diffusion: – equilibrates concentration differences
– preserves mass

Equilibration of Concentration Differences

Fick's law: $\mathbf{j} = -D \cdot \nabla u$
concentration gradient $\nabla u := (\partial_x u, \partial_y u)^\top$ creates *flux* $\mathbf{j} = (j_1, j_2)^\top$
D: *diffusion tensor* (symmetric, positive definite 2×2 matrix)

isotropic case: $\mathbf{j}$ parallel to ∇u
D degenerates to scalar-valued diffusivity
anisotropic case: $\mathbf{j}$ not parallel to ∇u

Conservation of Mass

continuity equation: $\partial_t u = -\mathrm{div}\,\mathbf{j}$ (t: time; $\mathrm{div}\,\mathbf{j} := \partial_x j_1 + \partial_y j_2$: divergence of $\mathbf{j}$).

The Diffusion Equation

Fick's law and continuity equation yield *diffusion equation*: $\partial_t u = \mathrm{div}\,(D\,\nabla u)$.
(also called heat equation)

Physical Background of Diffusion Processes (2)

Diffusion in Image Processing

- $u(x, y)$ denotes grey values instead of concentrations.
- D can be adapted (nonlinearly) to local image structure.

Three Important Cases

(a) *linear isotropic* diffusion filter with constant diffusivity
(Iijima, Witkin, Koenderink, Florack, Lindeberg, ...)

(b) *nonlinear isotropic* diffusion filters with scalar diffusivities being adapted the local image structure
(Perona/Malik, Catté et al., Whitaker, ...)

(c) *nonlinear anisotropic* diffusion filters with diffusion tensors being adapted to the local image structure
(Cottet, Weickert, Jawerth et al., Kimmel et al., ...)

Implications: $(c) \Longrightarrow (b) \Longrightarrow (a)$

Gaussian Convolution (1)

Gaussian Convolution

Let a grey-scale image be represented by a function $f \in L^1(\mathbb{R}^2)$, i.e.

$$\int_{\mathbb{R}^2} |f(\mathbf{x})|\, d\mathbf{x} \;<\; \infty \qquad (\mathbf{x} := (x, y)^\top).$$

A widely-used way to smooth f is to compute the *convolution ("Faltung")*

$$(K_\sigma * f)(\mathbf{x}) \;:=\; \int_{\mathbb{R}^2} K_\sigma(\mathbf{y})\, f(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$$

where K_σ is a 2-D *Gaussian* with standard deviation σ:

$$K_\sigma(\mathbf{x}) \;:=\; \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|\mathbf{x}|^2}{2\sigma^2}\right).$$

Its smoothing properties can be understood in a Fourier- and a PDE-based manner.

Gaussian Convolution (2)

Smoothing Properties in the Fourier Domain

The (complex-valued) *Fourier transformation*

$$(\mathcal{F}f)(\boldsymbol{\omega}) \;:=\; \int_{\mathbb{R}^2} f(\mathbf{x})\, \exp(-i\,\boldsymbol{\omega}^\top \mathbf{x})\, d\mathbf{x}$$

with $i := \sqrt{-1}$ decomposes an image $f(\mathbf{x})$ with $\mathbf{x} \in \mathbb{R}^2$ into its frequencies: $|(\mathcal{F}f)(\boldsymbol{\omega})|$ measures the content of a specific frequency $\boldsymbol{\omega} \in \mathbb{R}^2$.
The *convolution theorem* states that convolution becomes multiplication in the Fourier domain:

$$\left(\mathcal{F}(K_\sigma * f)\right)(\boldsymbol{\omega}) \;=\; (\mathcal{F}K_\sigma)(\boldsymbol{\omega}) \cdot (\mathcal{F}f)(\boldsymbol{\omega}),$$

The Fourier transform of a Gaussian is again Gaussian-shaped:

$$(\mathcal{F}K_\sigma)(\boldsymbol{\omega}) \;=\; \exp\left(-\frac{|\boldsymbol{\omega}|^2}{2/\sigma^2}\right).$$

Thus, Gaussian convolution attenuates high frequencies exponentially:
It is a low-pass filter.

Gaussian Convolution (3)

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Equivalence to Linear Diffusion Filtering

- Consider a linear diffusion process $u(\mathbf{x}, t)$ of a bounded image $f(\mathbf{x})$:

  $$\begin{aligned} \partial_t u &= \Delta u, \\ u(\mathbf{x}, 0) &= f(\mathbf{x}), \end{aligned}$$

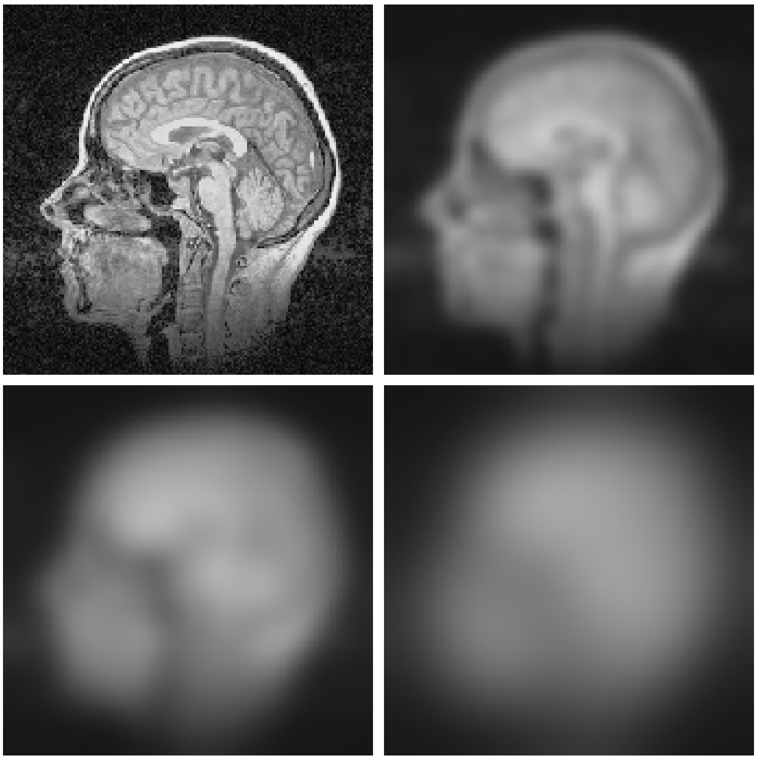
  where $\Delta u := \operatorname{div} \nabla u = \partial_{xx} u + \partial_{yy} u$ denotes the Laplacian.
- Books on PDEs show that it has the unique solution

  $$u(\mathbf{x}, t) = \begin{cases} f(\mathbf{x}) & (t = 0) \\ (K_{\sqrt{2t}} * f)(\mathbf{x}) & (t > 0) \end{cases}$$

  which depends continuously on the initial image f.
- solution $u(., t)$ is C^∞ in space for $t > 0$
- satisfies the *maximum–minimum principle*

  $$\inf_{\mathbb{R}^2} f \;\le\; u(\mathbf{x}, t) \;\le\; \sup_{\mathbb{R}^2} f \qquad \forall \mathbf{x},\, \forall t > 0.$$

- stopping time T is related to standard deviation σ via $T \;=\; \frac{1}{2}\sigma^2$



Linear diffusion filtering. (a) **Top left:** Original image, 236×236 pixels. (b) **Top right:** $t = 12.5$. (c) **Bottom left:** $t = 50$. (d) **Bottom right:** $t = 200$.

Gaussian Derivatives

Derivatives are useful for analysing local image structure (edges, corners), but:

Differentiation is Ill-Posed:

- A problem is called *well-posed ("gut gestellt")*, if it has a unique solution that depends continuously on the input data.
  If one of these conditions is violated, it is called *ill-posed ("schlecht gestellt")*.
- Example: A high-frequent 1-D perturbation $f(x) = \varepsilon \sin\left(\frac{x}{\varepsilon^2}\right)$ becomes arbitrarily small for $\varepsilon \to 0$. However, its derivative

$$f'(x) = \frac{1}{\varepsilon} \cos\left(\frac{x}{\varepsilon^2}\right)$$

  becomes unbounded!
- Thus, differentiation is ill-posed: Small perturbations in the image can create unbounded fluctuations in its derivatives.

Gaussian Derivatives (2)

Regularisation:

- For ill-posed problems one often uses *regularisation*:
  One embeds it in a family of well-posed problems with a parameter $\sigma > 0$.
  For $\sigma \to 0$, the original problem is recovered.
- Using some $\sigma > 0$ replaces the problem by a practically better tractable approximation.
- Sometimes already a discretisation may regularise a problem.
  However, then the solution may depend strongly on the type of discretisation.

Possibility to Regularise Differentiation:

- apply Gaussian convolution before differentiation
- equivalent to convolution with derivatives of the Gaussian:

$$\partial_{x_1}^n \partial_{x_2}^m (K_\sigma * f) \;=\; (\partial_{x_1}^n \partial_{x_2}^m K_\sigma) * f$$

- resulting so-called *Gaussian derivatives* are bounded

Gaussian Derivatives (3)

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Gaussian derivatives ($\sigma = 1$). (a) **Top left:** Original image, 256×256 pixels. (b) **Top middle:** x derivative. (c) **Top right:** y derivative. (d) **Bottom left:** Gradient magnitude. (e) **Bottom right:** Laplacian.

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Edge Detection with Gaussian Derivatives

Gradient

- Gradient magnitude of $f_\sigma := K_\sigma * f$: $$|\nabla f_\sigma| := \sqrt{(\partial_x f_\sigma)^2 + (\partial_y f_\sigma)^2}$$
- edges: locations where $|\nabla f_\sigma|$ is large
- quite robust; fuzzy; does not give closed contours

Laplacian

- *Laplacian-of-Gaussian (LoG)*: $$\Delta f_\sigma := \partial_{xx} f_\sigma + \partial_{yy} f_\sigma$$
- edges: zero-crossings of Δf_σ
- less robust; non-fuzzy; gives closed contours
- **interesting observation:** (Witkin 1983)
  - zero-crossings can be traced back to finer scales *(causality)*
  - equivalent to maximum-minimum principle (Hummel 1986)
  - has led to the *scale-space* formulation in the western world

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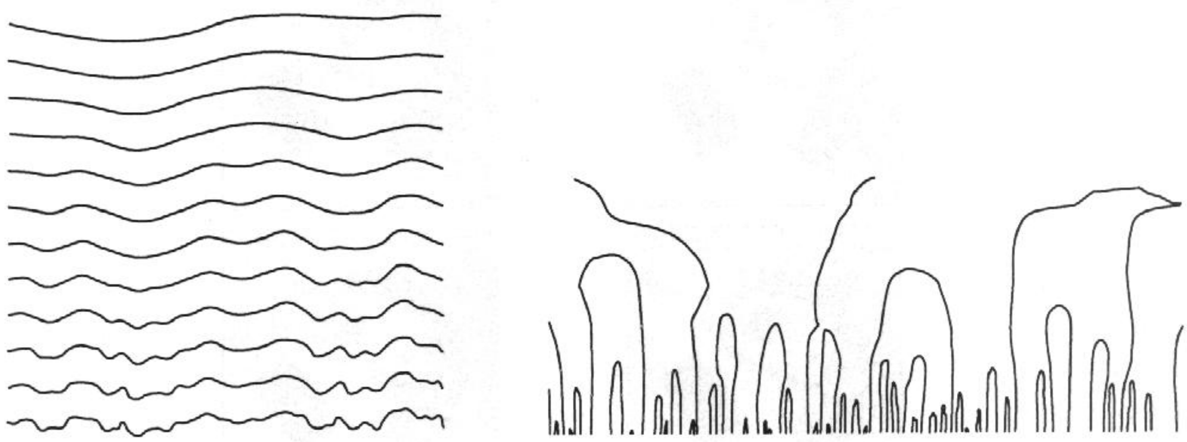
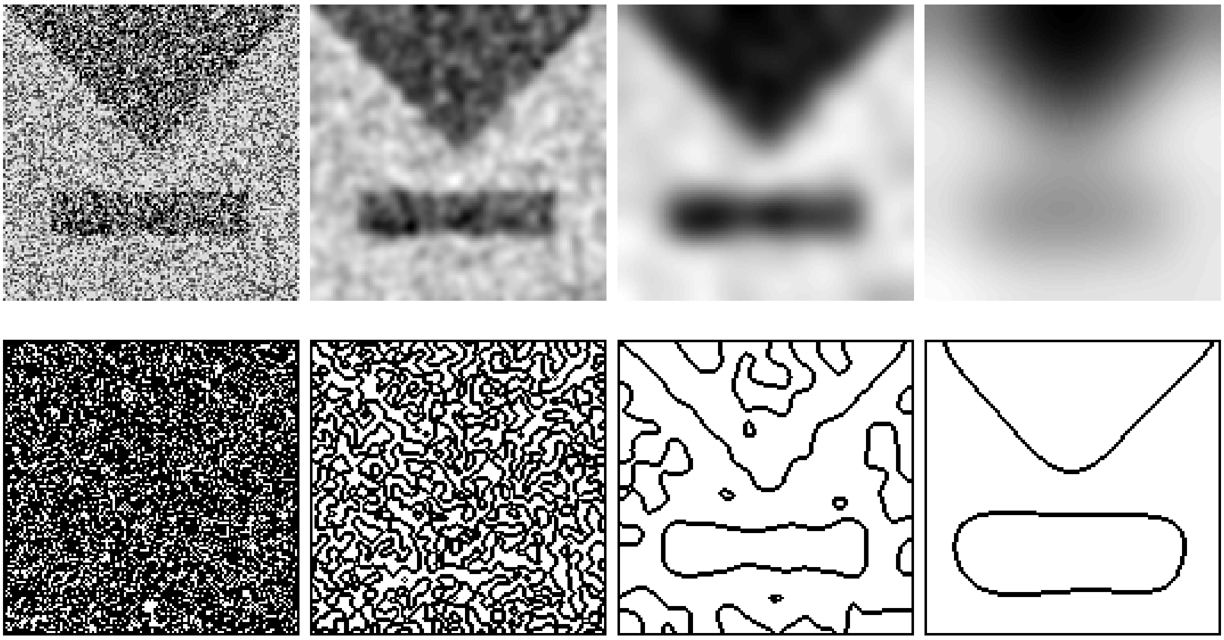


(a) **Left:** Evolution of a signal in Gaussian scale-space. The standard deviation σ is increasing from bottom to top. (b) **Right:** Corresponding evolution of the zero-crossings of the Laplacian. The vertical axis denotes scale, the horizontal axis describes the location. **Authors:** Tony Lindeberg and Bart ter Haar Romeny (1994). Adapted from A. P. Witkin: Scale-space filtering. *Proc. Eighth Int. Joint Conf. on Artificial Intelligence* (IJCAI '83, Karlsruhe, Aug. 8–12, 1983), Vol. 2, 1019–1022, 1983.

Edge Detection with Gaussian Derivatives (3)



(a) **Top:** Evolution of a noisy test image (128×128 pixels) in Gaussian scale-space. From left to right: Standard deviation of Gaussian: $\sigma = 0,\ 2,\ 6,\ 18$. (b) **Bottom:** Corresponding zero-crossings of the Laplacian.

The Scale-Space Concept (1)

The Scale-Space Concept

Motivation

- features exist only at a certain range of scales;
  if scale unclear, multiscale description reasonable
- hierarchy of image features

Scale-Space

- embedding of an image $f : \mathbb{R}^2 \to \mathbb{R}$ into an ordered family

$$\{T_t f \mid t \geq 0\}$$

of gradually smoothed, simplified versions that satisfies certain requirements.

The Scale-Space Concept (2)

Scale-Space Requirements

Alvarez *et al.* (1993) distinguish three classes of requirements:

- *architectural properties,*
  e.g. the semi-group property

$$T_{t+s}f \;=\; T_t(T_s f) \qquad \forall\, s,t \geq 0.$$

- *simplification properties,*
  e.g. causality, extremum principles

- *invariances,*
  e.g. translation and rotation invariance.

Gaussian Scale-Space (1)

Gaussian Scale-Space

- oldest and best-studied scale-space
- computes scale-space evolution as convolution of Gaussians with increasing size:

$$T_t f \;=\; K_{\sqrt{2t}} * f \;:=\; \int_{\mathbb{R}^m} K_{\sqrt{2t}}(\mathbf{y})\, f(\mathbf{x}-\mathbf{y})\, d\mathbf{y}$$

Historical Aspects (Weickert/Ishikawa/Imiya 1999):

- Usually Witkin (1983) is considered as the "inventor" of Gaussian scale-space.
- However, it has already been axiomatically derived by Taizo Iijima in 1962.
- entire world of early Japanese scale-space research that remained unknown in the western world
- English translations have been available for many of these papers
- applications to optical character recognition (OCR)

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Gaussian Scale-Space (2)

UDC 007

パターンの正規化に関する基礎理論

（典型的な 1 次元パターンの場合）

Basic Theory on Normalization of Pattern

(In Case of Typical One-Dimensional Pattern)

(1962, 4, 26 受付)

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Taizo IIJIMA

パターンを正規化する問題はパターン認識における最も基本的な問題の一つであるが，本論文はこの問題が観測理論の立場から解決されることを理論的に示したものである。本論文では理解し易いために典型的な一次元パターンの場合を考察した。まずパターンという概念を観測の立場から明確化することによつて，観測機構の持つべき機能形式を理論的に導出し，特に視覚機構がレンズ系によつて実現される理由を明らかにした。次いでこのような観測形式にもとづいて観測されるパターンの本質的情報とは何であるかを解明し，かような情報を観測像から抽出する数学的方法を確立した。この結果はパターンの正規化法を与えるものである。本論文は文字や音声などに関するパターンの正規化法を確立するために，理論展開の原型を与えようとしたものである。

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Summary

Summary

- Diffusion is a physical process that equilibrates concentration differences and preserves mass. Its PDE is given by $\partial_t u = \operatorname{div}(D\,\nabla u)$
- For $D = I$ one obtains the linear diffusion equation $\partial_t u = \Delta u$.
- Its solution is given by Gaussian convolution.
- comes down to multiplication with a Gaussian in the Fourier domain
- relation between diffusion time t and kernel width σ:

$$t = \frac{1}{2}\sigma^2$$

- Gaussian derivatives are useful for edge detection
(gradient magnitude or zero-crossings of the Laplacian).
- The zero-crossings of the Laplacian obey a causality principle.
- Gaussian smoothing creates a scale-space.
- has already been discovered in Japan by Taizo Iijima in 1962.

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References

- J. Weickert: *Anisotropic Diffusion in Image Processing*. Teubner, Stuttgart, 1998.
  (This lecture is based on Sections 1.1 and 1.2.)
- H. S. Carslaw, J. C. Jaeger: *Conduction of Heat in Solids.* Oxford University Press, Second Edition, 1959.
  (still one of the best monographs on the physics of heat conduction and diffusion)
- J. Crank: *The Mathematics of Diffusion*. Oxford University Press, Second Edition, 1975.
  (another classical book on diffusion)
- L. Alvarez, F. Guichard, P.-L. Lions, J.-M. Morel: Axioms and fundamental equations in image processing. *Archive for Rational Mechanics and Analysis*, Vol. 123, 199–257, 1993.
  (axiomatic approach to linear and nonlinear scale-spaces)
- A. P. Witkin: Scale-space filtering. *Proc. Eighth International Joint Conference on Artificial Intelligence*, (Karlsruhe, Germany, Aug. 1983), Vol. 2, 945–951, 1983.
  (first scale-space paper in the western world)
- J. Weickert, S. Ishikawa, A. Imiya: Linear scale-space has first been proposed in Japan, *J. Math. Imag. Vision*, Vol. 10, 237–252, 1999.
  (`www.mia.uni-saarland.de/weickert/Papers/diku9718.ps.gz`)
  (a survey on early Japanese contributions to Gaussian scale-space analysis)
- L. Florack, *Image Structure*, Kluwer, Dordrecht, 1997.
  (applies many concepts from theoretical physics and differential geometry)

- T. Lindeberg, *Scale-Space Theory in Computer Vision*. Kluwer, Boston, 1994.
  (monograph on Gaussian scale-space theory with emphasis on scale selection concepts)
- J. Sporring, M. Nielsen, L. Florack, P. Johansen (Eds.): *Gaussian Scale-Space Theory*. Kluwer, Dordrecht, 1997.
  (a collection that covers many aspects ranging from axiomatics and singularity theory to applications)

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Assignment T1 – Theoretical Home Work

Problem 1 (Convolution, Derivatives and Diffusion Equation) (7 points)

(a) Prove that for continuously differentiable functions $f(x)$ and $g(x)$ the following equality holds for all $x \in \mathbb{R}$:

$$(f * g)'(x) \;=\; (f' * g)(x) \;=\; (f * g')(x).$$

(b) Verify that

$$u(x,t) = K_{\sqrt{2t}}(x)$$

is a solution of the linear diffusion equation $u_t = u_{xx}$ for $x \in \mathbb{R}$, $t > 0$.

(c) Prove that

$$u(x,t) = (K_{\sqrt{2t}} * f)(x)$$

is a solution of the initial value problem

$$\begin{aligned} u_t &= u_{xx}, \quad & x \in \mathbb{R}, \quad t > 0 \\ u(x,0) &= f(x), \quad & x \in \mathbb{R} \end{aligned}$$

for any given bounded continuous function f.
You can use that $K_0 * f = f$.

Problem 2 (Numerical Differentiation) (5 points)

For a smooth function f of one variable, let the values f_i at the discrete points $x = i$, i integer, be given.
Find a finite difference approximation for the second derivative f'' of highest possible consistency order in the point i using the function values f_{i-2}, f_{i-1}, f_i, f_{i+1}, f_{i+2}. Determine this consistency order.

Deadline for submission: Friday, April 25, 10 am (before the lecture).