Differential Equations in Image Processing and Computer Vision

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www.mia.uni-saarland.de/Teaching/dic08.shtml

Organisatorial Issues

Welcome to this class!
Please do not hesitate to pose questions anytime (in English or German).

What Type of Lectures is It ?

- 4 hours classroom lectures, 2 hours tutorials (9 ECTS points)
- can be used in different ways:
  - *visual computing:* core area visual computing, image analysis class
  - *computer science:* advanced class (Vertiefungsvorlesung)
  - *mathematics:* applied mathematics class
- qualify for starting a master or diploma thesis in our group

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Organisatorial Issues (2)

What Prerequisites are Required?

- undergraduate mathematics (e.g. Mathematik für Informatiker I–III)
- elementary C knowledge (for the programming assignments)
- passive knowledge of English
  (solutions in assigments or exam can be submitted in German)
- Specific knowledge on partial differential equations, image processing or computer vision is not required (but it helps of course).

Tutorials

- theoretical and programming assignments, alternating each week
- three groups planned: Tue 16–18, Thu 12–14, Thu 16–18
- coordination of the tutorials: Stephan Didas
- online registration for tutorials: Tue, Apr. 15, 2:00 pm to Fri, Apr. 18, 4:00 pm.
- start: next week
- more details on the web page

Organisatorial Issues (3)

Grading Policy

- Bilingual exam on July 22, 2:00–5:00 pm (E2.5, Lecture Hall 1).
  Second chance on October 14, 2:00–5:00 pm (E2.5, Lecture Hall 1).
  The better grade counts.
- In order to qualify for the exam you must
  - attend 80 % of the programming and the theoretical tutorials (we do check).
  - solve 50 % of the all assignments (theory and programming) correctly.
    Working in groups of up to 3 people is permitted, but all persons must be in the same tutorial group.

Registration and Lecture Notes

- Please register for your tutorial group as of today, April 15, 2:00 pm:

  `www.mia.uni-saarland.de/Teaching/dic08.shtml`

  Closing date: Friday, April 18, 4:00 pm
- You can also find the password-protected lecture notes there.

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Contents and Goals

What is It All About ?

- Partial differential equations (PDEs) are used in some of the best performing image processing and computer vision methods.
- often apply ideas from physics (e.g. diffusion, wave propagation) in order to solve image analysis problems
- applications include image denoising, enhancement, deblurring, multiscale representations, motion analysis, stereo reconstruction, shape-from-shading
- belong to the mathematically best-founded image analysis methods
- closely related to optimisation approaches from the calculus of variations
- in contrast to many other methods, *continuous* models are used; adequate rotationally invariant models
- have led to the unification of a number of existing approaches, as well as to the discovery of novel methods

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After This Class You Should be Able

- to understand the basic ideas behind the most important methods, in particular their advantages and shortcomings
- to find an appropriate method for a task at hand
- to understand the basic algorithms
- to start a master or diploma thesis in our group

Literature

- J. Weickert: *Anisotropic Diffusion in Image Processing.* Teubner, Stuttgart, 1998.
  Out of print, but predecessing Ph.D. thesis available from
  `http://www.mia.uni-saarland.de/weickert/Papers/diss.ps.gz`
  (main reference, in particular for the diffusion part)
- G. Aubert and P. Kornprobst: *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations.* Springer, New York, 2002 (Second Edition: 2006).
  (excellent for variational models and mathematically more advanced results)
- F. Cao: *Geometric Curve Evolution and Image Processing.* Lecture Notes in Mathematics, Vol. 1805, Springer, Berlin, 2003.
  (nice book on curve evolutions; inexpensive)
- R. Kimmel: *The Numerical Geometry of Images.* Springer, New York, 2004.
  (emphasises curve evolution aspects and his so-called Beltrami framework)
- G. Sapiro: *Geometric Partial Differential Equations in Image Analysis.* Cambridge University Press, 2001.
  (essentially a collection of Sapiro's publications on curve evolution)
- Many articles from journals and conferences.

The main books are available at the computer science / applied mathematics library (Semesterapparat).
Each lecture provides specific references. Please ask me if you have no access to them.

Lecture 1:
An Overview of PDEs in Image Analysis

Contents

What are PDEs? (1)

What are PDEs ?

- *Algebraic equations* state relations between an unknown *number* and its powers, e.g.

$$x^2 - 8x + 15 = 0.$$

Two solutions: $x_1 = 3$, $x_2 = 5$.

- *Differential equations* state relations between an unknown *function* and its derivatives, e.g.

$$\frac{du(t)}{dt} = 5u(t).$$

Infinitely many solutions:

$$u(t) = a\ \exp(5t) \qquad (a \in \mathbb{R}).$$

An additional *initial condition*, e.g. $u(0) = 2$, can make the solution unique:

$$u(t) = 2\exp(5t).$$

If u depends only on one variable (t), we have ordinary derivatives, and the differential equation is an *ordinary differential equation (ODE)*.

What are PDEs? (2)

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- If u depends on multiple variables, e.g. x and t, then partial derivatives may occur: *partial differential equation (PDE)*.

Example: Consider the so-called diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with the initial state $f(x)$,

$$u(x, t=0) = f(x).$$

Books on PDEs tell us that this problem has a Gaussian-smoothed version of $f(x)$ as solution:

$$u(x,t) = (K_{\sqrt{2t}} * f)(x) := \int_{-\infty}^{\infty} K_{\sqrt{2t}}(y)\, f(x-y)\, dy$$

K_σ: Gaussian with standard deviation σ

Such nice analytical solutions are only known for the simplest PDEs.
Usually, numerical approximations are necessary.

Diffusion Processes

Scale-space:

embeds an image $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ into a family $\{T_t f \mid t \geq 0\}$ of gradually smoothed, simplified versions

Scale-spaces are governed by PDEs:

The simplified image $T_t f$ is the solution $u(t)$ of

$$\partial_t u \;=\; F(\nabla u, \mathrm{Hess}(u))$$

with the original image f as initial condition:

$$u(t\!=\!0) \;=\; f$$

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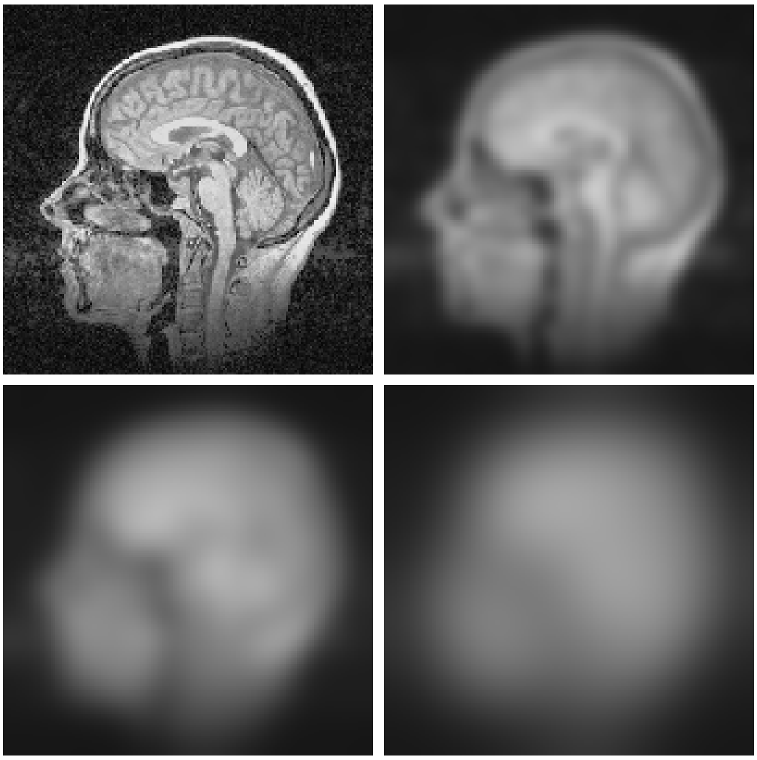
Example: Gaussian scale-space

$$T_t f \;=\; K_{\sqrt{2t}} \, * \, f$$

$T_t f$ can be shown to be identical to the solution $u(x, y, t)$ of linear diffusion filtering:

$$\begin{aligned} \partial_t u &= \partial_{xx} u + \partial_{yy} u \\ u(t\!=\!0) &= f \end{aligned}$$

Diffusion Processes (3)



Scale-space behaviour of linear diffusion filtering. (a) **Top left:** Original image, 236×236 pixels. (b) **Top right:** $t = 12.5$. (c) **Bottom left:** $t = 50$. (d) **Bottom right:** $t = 200$.

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Diffusion Processes (4)

The scale-space concept solves two fundamental questions:

- **On which scales are important structures ?**
  - scale selection criteria, focus-of-attention:
    features that exist on scale t_1 but not on the larger scale t_2, belong to the scale range $[t_1, t_2)$.
- **How are the structures organised ?**
  - stability over scales measures importance
  - scale-space gives hierarchy of features
  - multiscale segmentation:
    identify a structure at some coarse scale (where noise has been removed) and trace it back to some finer scale (where the location is more precise).

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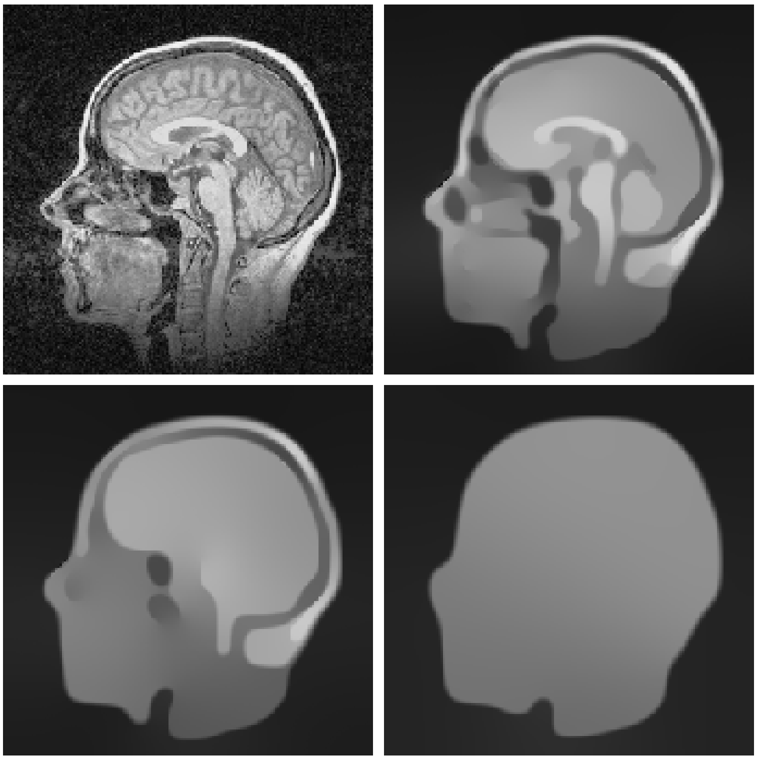
Diffusion Processes (5)

Nonlinear Diffusion: Basic Idea

- steer the activity and the direction of the diffusion process by the features of the evolving image
- e.g. for edge-preserving smoothing or closing of interrupted lines

Diffusion Processes (6)



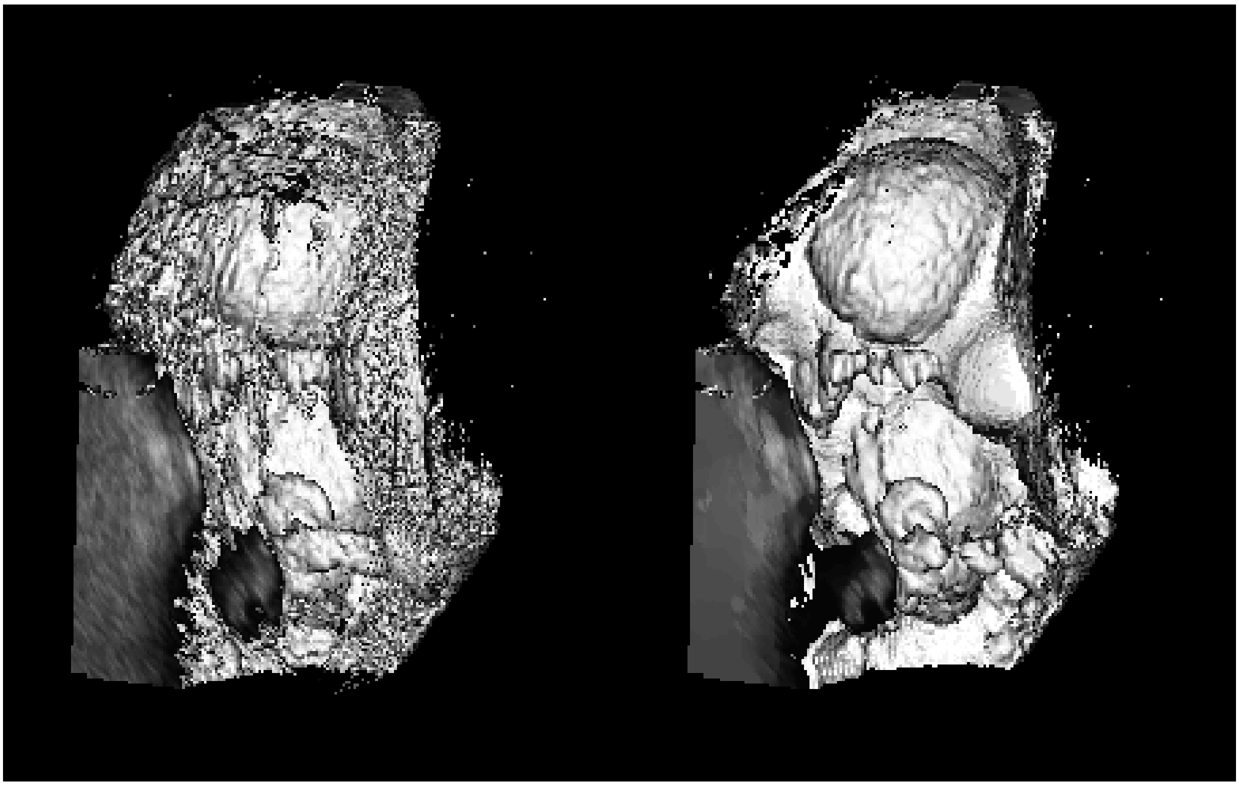
Scale-space behaviour of a nonlinear diffusion filter. (a) **Top left:** Original image, 236×236 pixels. (b) **Top right:** $t = 250$. (c) **Bottom left:** $t = 875$. (d) **Bottom right:** $t = 3000$.

Diffusion Processes (7)

Nonlinear Diffusion: Goals

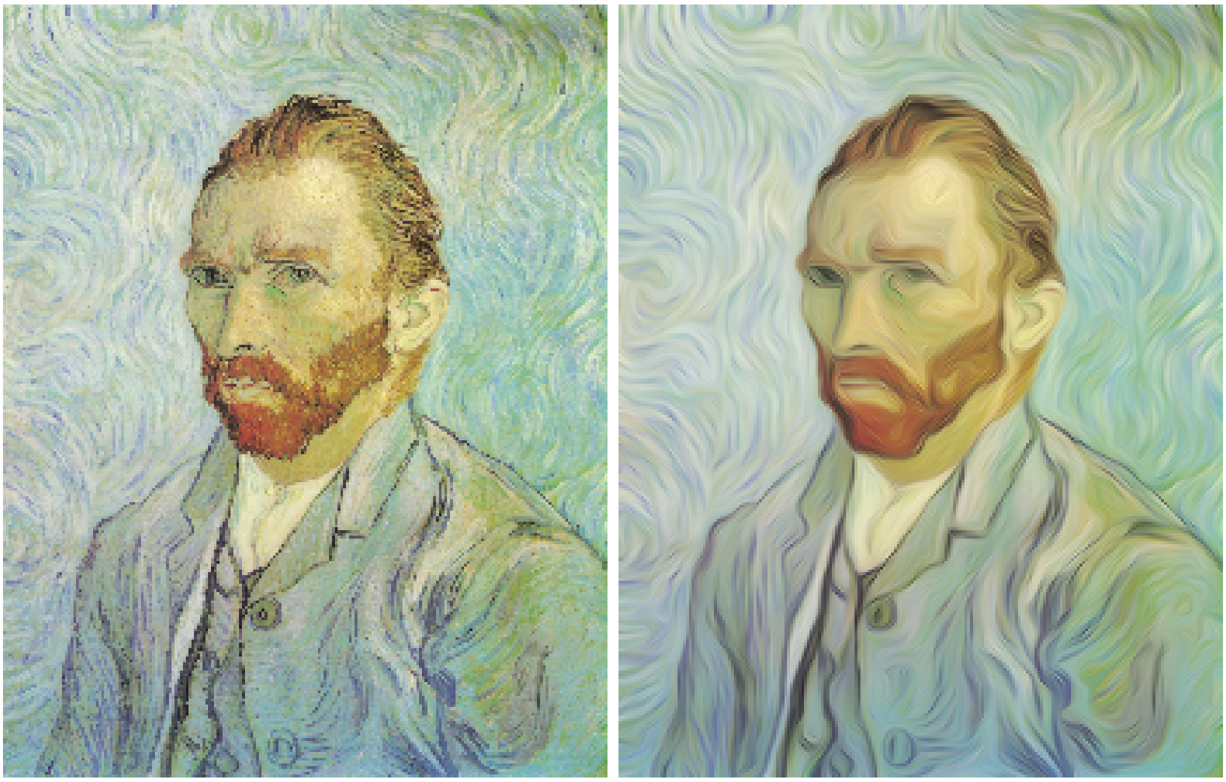
- investigate suitable diffusion filters for a task at hand
- study their continuous and discrete properties
- find design criteria for reliable algorithms
- study possibilities to speed-up algorithms by better numerical schemes and parallel implementations
- extend these models to colour images and even matrix-valued images
- find connections to other methods such as wavelet-based filters
- use diffusion processes for restoring incomplete images (interpolation, inpainting)

Diffusion Processes (8)



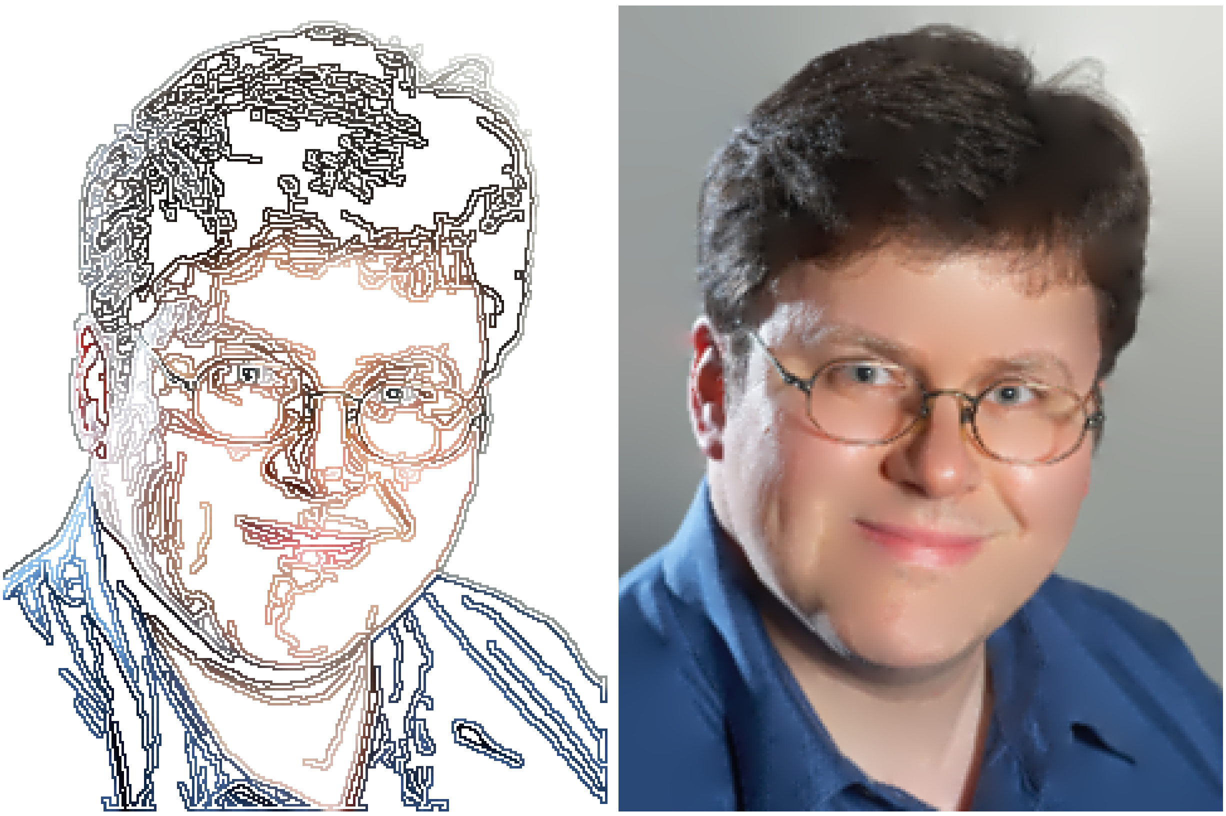
Rendering of a 3-D ultrasound image of a 10-week old fetus. **Left:** Original data, $138 \times 208 \times 138$ voxels. **Right:** Filtered with a fast parallel algorithm for nonlinear diffusion.

Diffusion Processes (9)



(a) **Left:** Self portrait by van Gogh (Saint-Rémy, 1889; Paris, Museé d'Orsay). 215×275 pixels. (b) **Right:** Filtered with a nonlinear diffusion process that enhances flow-like structures.

Diffusion Processes (10)



Left: Sparse representation using image edges. **Right:** Reconstruction by a diffusion process.

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Image Sequence Analysis

Problem:

- given: image sequence $f(x, y, \theta)$, where θ denotes time
- wanted: displacement field between corresponding pixels in subsequent frames: *optic flow* $\binom{u(x,y,\theta)}{v(x,y,\theta)}$
- optic flow is an example for a key problem in computer vision:
  - find correspondences between two images
  - also important for stereo reconstruction, medical image registration, ...

Solution:

- modelled via a continuous optimisation framework
- solution can be found using PDEs that resemble those for diffusing colour images

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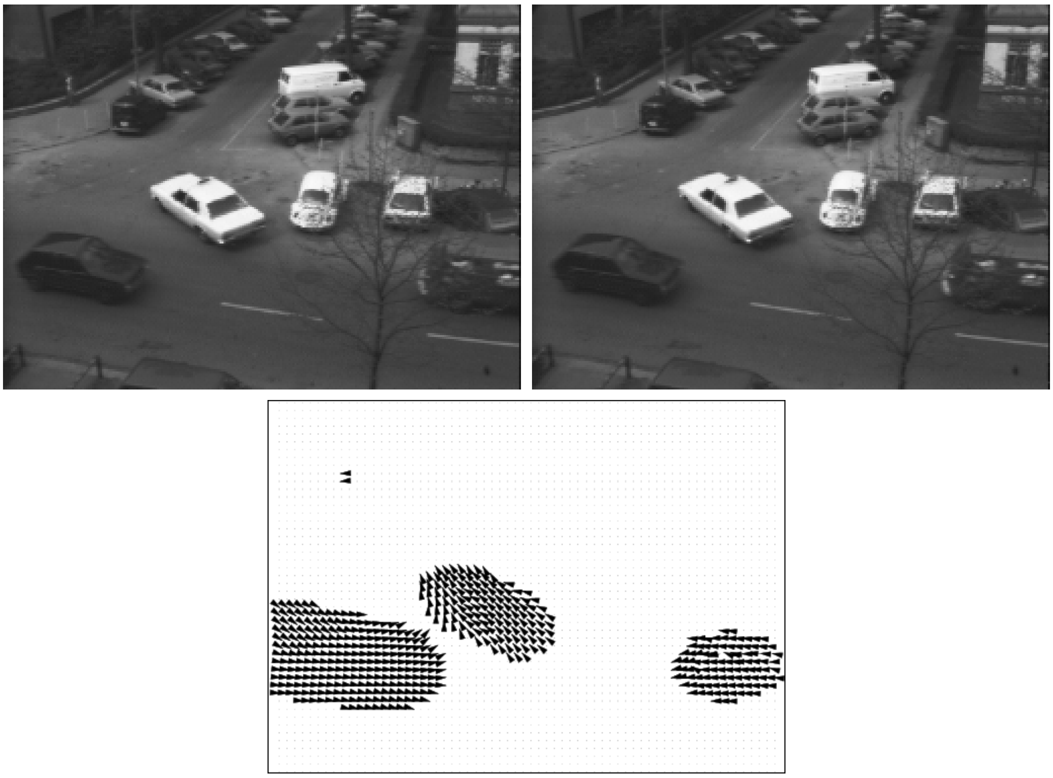
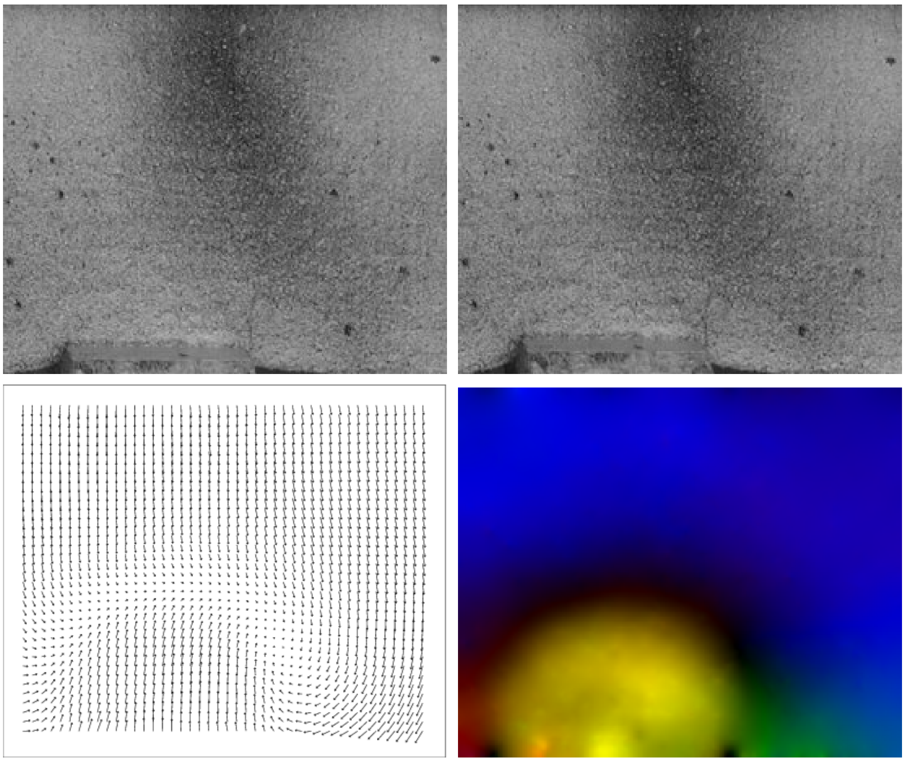


(a) **Top left:** Frame 10 of the Hamburg taxi sequence. (b) **Top right:** Frame 11. (c) **Bottom:** Optic flow field.



Deformation analysis of plastic foam using an optic flow method. (a) **Top left:** Frame 1 of a deformation sequence, 273×224 pixels. (b) **Top right:** Frame 2. (c) **Bottom left:** Vector plot of the displacement field. (d) **Bottom right:** Colour-coded displacement field.

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Classical Morphological Processes

Morphology:

- analysis of shapes
- two basic operations: dilation, erosion
- dilation/erosion with a disk of radius t replaces each greyvalue by its maximum/minimum within a disk of radius t
- can be computed as solution $u(t)$ of

$$\partial_t u \;=\; \pm|\nabla u|$$

  with original image f as initial condition.
- Combining dilation and erosion in a suitable way can lead to very interesting filters.

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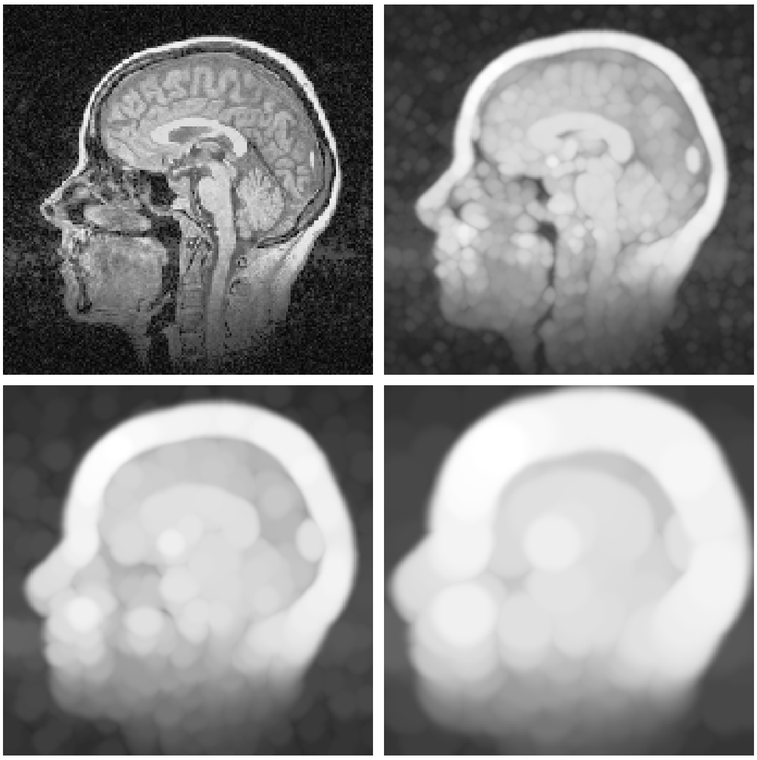
Classical Morphological Processes (2)



Scale-space behaviour of dilation with a disk. (a) **Top left:** Original image, 236×236 pixels. (b) **Top right:** $t = 4$. (c) **Bottom left:** $t = 10$. (d) **Bottom right:** $t = 20$.

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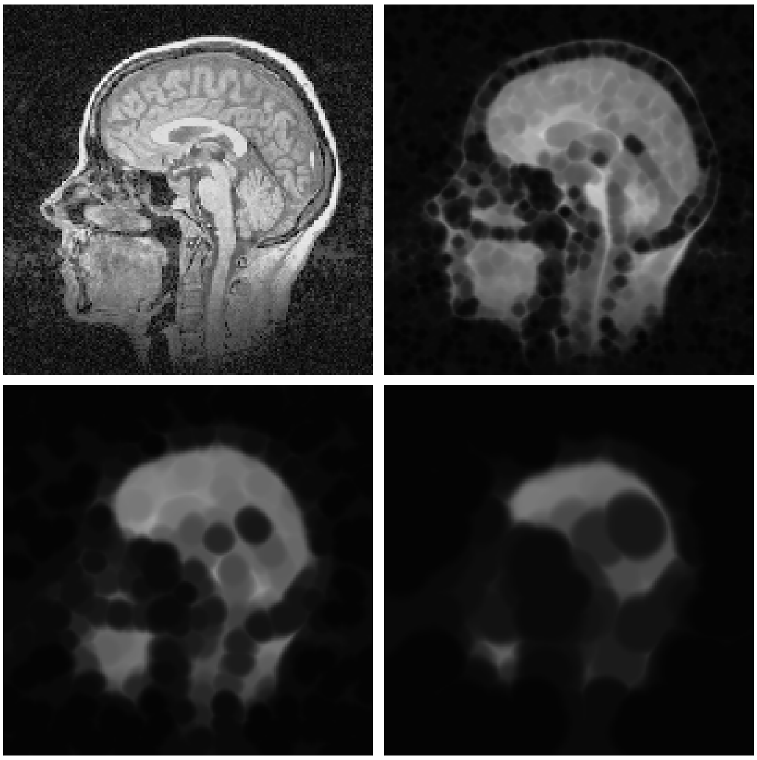
Classical Morphological Processes (3)



Scale-space behaviour of erosion with a disk. (a) **Top left:** Original image, 236×236 pixels. (b) **Top right:** $t = 4$. (c) **Bottom left:** $t = 10$. (d) **Bottom right:** $t = 20$.

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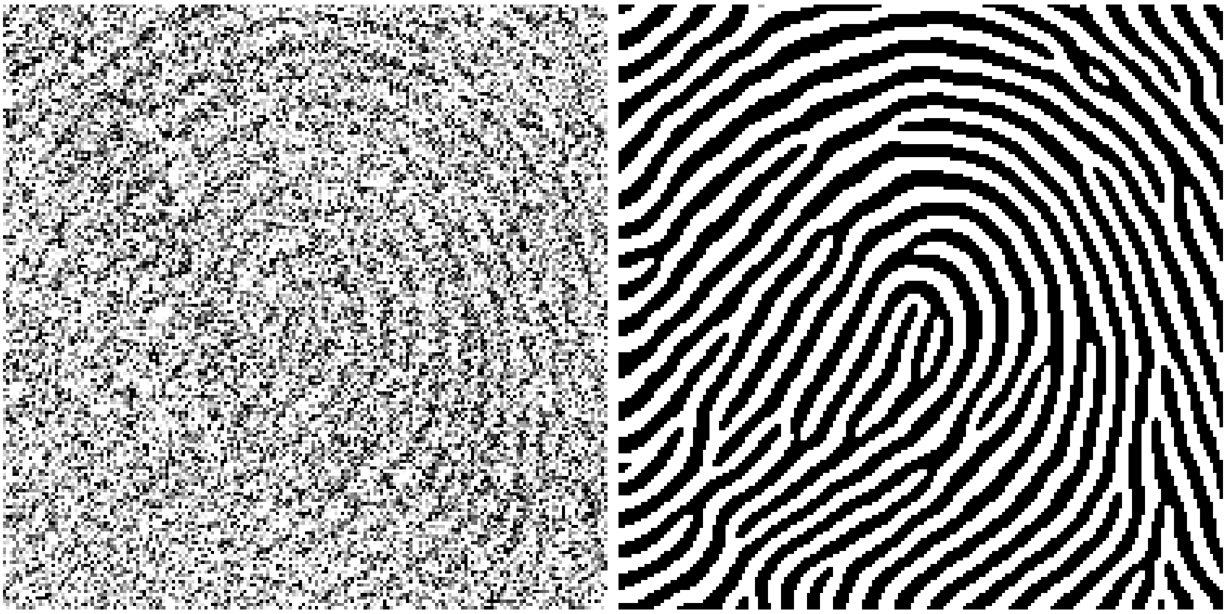

(a) **Left:** Fingerprint image with severe noise, 186×186 pixels. (b) **Right:** Restoration with a method that performs locally either dilation or erosion, depending on the underlying image structure.

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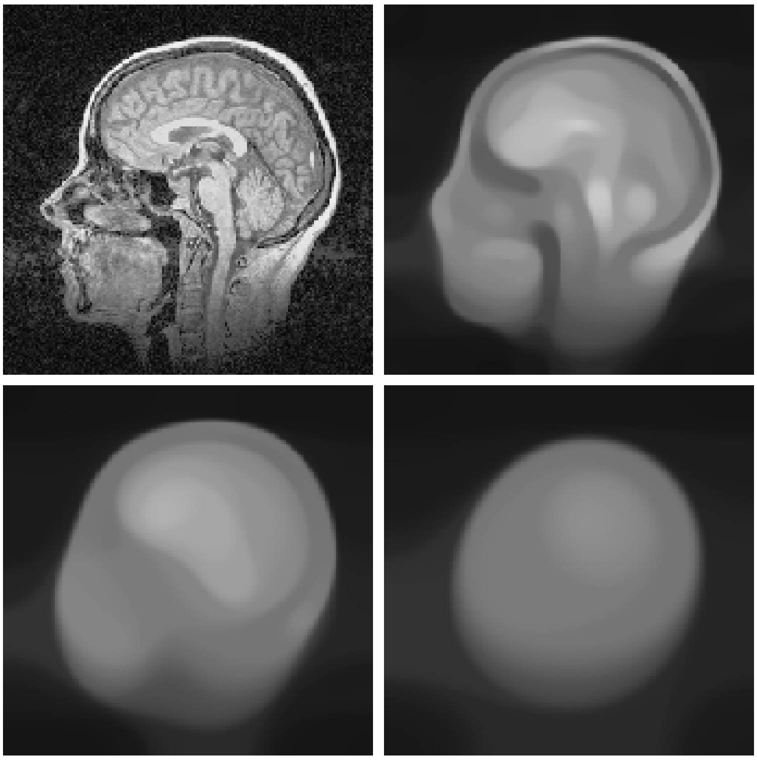
Curvature-Based Morphology

- simplification of the level lines of images
- depending on the local curvature of the level lines, either dilation or erosion takes place
- example: mean curvature motion

$$\partial_t u \;=\; |\nabla u| \, \mathrm{curv}(u).$$

- moves level lines in normal direction with a speed proportional to their curvature
- level lines become more and more circular and shrink in finite time to a point
- modifications of this process can be used for *snakes*:
  a user-specified active contour moves to an object of interest

Curvature-Based Morphology (2)



Mean curvature motion. (a) **Top left:** Original image, 236×236 pixels. (b) **Top right:** $t = 70$. (c) **Bottom left:** $t = 275$. (d) **Bottom right:** $t = 1275$.

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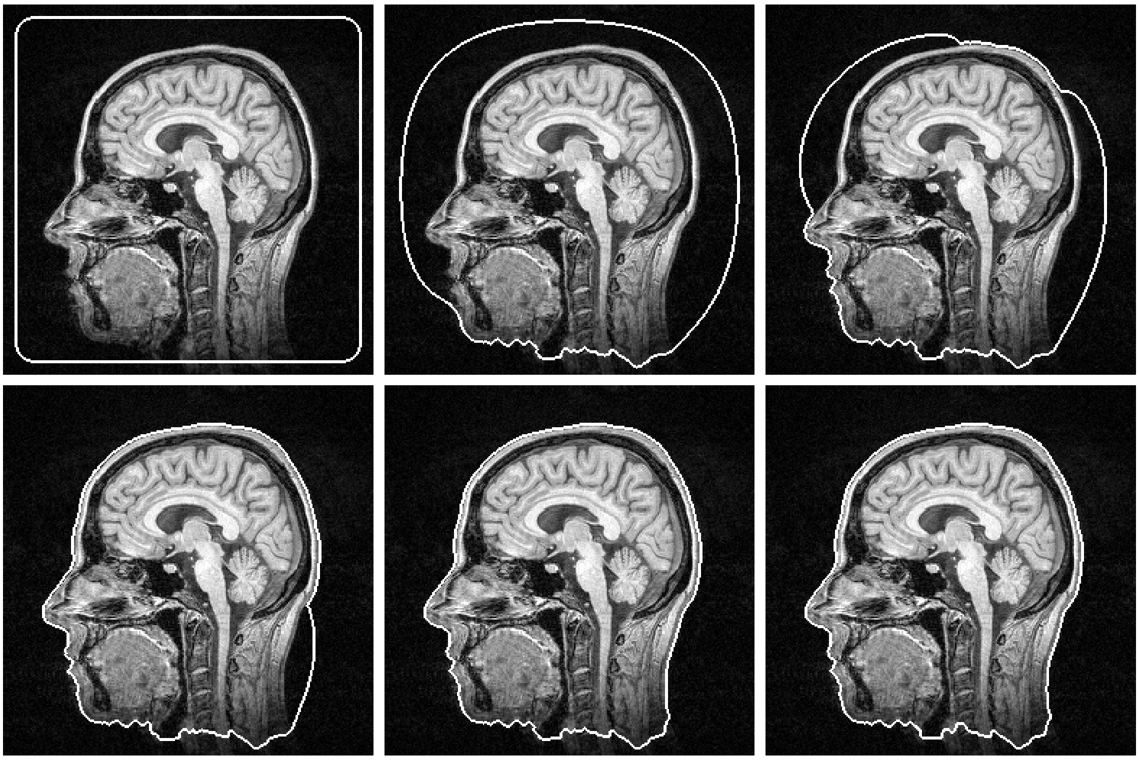
Curvature-Based Morphology (3)



Temporal evolution of a geodesic active contour model. **From top left to bottom right:** $t = 0$, 1500, 3000, 4500, 6000, 7500.

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Summary

- PDEs are useful and mathematically well-founded concepts for scale-space analysis, image enhancement, for solving correspondence problems and for shape analysis.
- allow a reinterpretation of classical methods (such as Gaussian smoothing or dilation/erosion) under a unifying framework
- have also led to new methods with interesting properties (nonlinear diffusion filters, PDE-based optic flow methods, mean curvature motion, active contours)
- continuous formulation leads to approximations with good rotational invariance.

References

Some Literature on PDEs

- S. J. Farlow, *Partial Differential Equations for Scientists and Engineers*. Dover, New York, 1993. *(well-readable introduction for non-mathematicians)*.
- L. C. Evans, *Partial Differential Equations*. American Mathematical Society, Providence, 1998. *(state-of-the-art monograph on the modern theory of linear and nonlinear PDEs)*
- K. W. Morton, D. F. Mayers, *Numerical Solution of Partial Differential Equations*. Cambridge University Press, Cambridge, 1994. *(well readable introduction to finite difference methods for the numerical solution of (mostly linear) PDEs)*

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