Differential Equations in Image Processing and Computer Vision

Assignment T6

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Group 2: Thu, 12-14 (Markus Mainberger)

1	2	3	4	$\sum$

6.1 Discretisation of Hyberbolic PDEs

a.

- forward differences:

$$\begin{aligned} \frac{u_i^{k+1} - u_i^k}{\tau} &= a \cdot \frac{u_{i+1}^k - u_i^k}{h} \\ \Leftrightarrow u_i^{k+1} &= a\tau \cdot \frac{u_{i+1}^k - u_i^k}{h} + u_i^k \end{aligned}$$

- backward differences:

$$\begin{aligned} \frac{u_i^{k+1} - u_i^k}{\tau} &= a \cdot \frac{u_i^k - u_{i-1}^k}{h} \\ \Leftrightarrow u_i^{k+1} &= a\tau \cdot \frac{u_i^k - u_{i-1}^k}{h} + u_i^k \end{aligned}$$

- central differences:

$$\begin{aligned} \frac{u_i^{k+1} - u_i^k}{\tau} &= a \cdot \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{2h} \\ \Leftrightarrow u_i^{k+1} &= a\tau \cdot \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{2h} + u_i^k \end{aligned}$$

b. The results from above should now be represented as a convex combination.

- forward differences:

$$u_i^{k+1} = \frac{a\tau}{h} u_{i+1}^k + \left(1 - \frac{a\tau}{h}\right) u_i^k$$

To fullfill one criterion of the convex combination $\frac{a\tau}{h} > 0$ and since $\tau, h > 0$ it must hold that $a > 0$. For the other part of the convex combination you see that

$$\begin{aligned} 1 - \frac{a\tau}{h} &> 0 \\ \Leftrightarrow \frac{a\tau}{h} &< 1 \\ \Leftrightarrow a &< \frac{h}{\tau} \end{aligned}$$

If $0 < a < \frac{h}{\tau}$, the max-min-stability holds.

- backward differences:

$$u_i^{k+1} = \left(1 + \frac{a\tau}{h}\right) u_i^k - \frac{a\tau}{h} u_{i-1}^k$$

In this case, on the one hand side it must be fullfilled that $-\frac{a\tau}{h} > 0 \Leftrightarrow \frac{a\tau}{h} < 0$. On the other hand it must hold the following:

$$\begin{aligned} 1 + \frac{a\tau}{h} &> 0 \\ \frac{a\tau}{h} &> -1 \\ a &> -\frac{h}{\tau} \end{aligned}$$

If $-\frac{h}{\tau} < a < 0$, the max-min-stability holds.

- central differences:

$$u_i^{k+1} = \left(1 - 2\frac{a\tau}{h}\right) u_i^k + \frac{a\tau}{h}(u_{i+1}^k + u_{i-1}^k)$$

In this case, $\frac{a\tau}{h} > 0 \Leftrightarrow a > 0$. Furthermore

$$\begin{aligned} 1 - 2\frac{a\tau}{h} &> 0 \\ 2\frac{a\tau}{h} &< 1 \\ a &< \frac{h}{2\tau} \end{aligned}$$

If $0 < a < \frac{h}{2\tau}$, the max-min-stability holds.

6.2 Slope Transform

We know that

$$\mathcal{S}[f](u) := \operatorname*{stat}_x (f(x) - ux)$$

Since $\operatorname*{stat}_x g(x) := g(x) | g'(x) = 0$ we can plug this in:

$$\begin{aligned}
\mathcal{S}[f_p](u) &= \operatorname*{stat}_x (f_p(x) - ux) \\
&= \left\{ f_p(x) - ux \mid f_p'(x) - u = 0 \right\} \\
&= \left\{ cx^p - ux \mid pcx^{p-1} - u = 0 \right\} \\
&= \left\{ x\left(cx^{p-1} - u\right) \;\middle|\; x^{p-1} = \frac{u}{pc} \right\} \\
&= \left\{ x\left(\frac{u}{p} - u\right) \;\middle|\; x = \operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}} \right\} \\
&= \left\{ \operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}} \left(\frac{u}{p} - u\right) \right\} \\
\mathcal{S}[f_p](u) &= \left\{ \operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}} \frac{u}{p}(1-p) \right\}
\end{aligned}$$

For the case that $p = 2$ we get the following:

$$\begin{aligned}
\mathcal{S}[f_2](u) &= \left\{ cx^2 - ux \mid 2cx - u = 0 \right\} \\
&= \left\{ x(cx - u) \mid x = \frac{u}{2c} \right\} \\
&= \left\{ \frac{u}{2c}\left(\frac{u}{2} - u\right) \right\} \\
&= \left\{ -\frac{u^2}{4c} \right\}
\end{aligned}$$

In this case the slope transform remains a quadratic function. This shows the fact that parabolas remain parabolas (to be more precise "paraboloids") under the slope transform (Lecture 21, slide 13).

6.3 Curvature-Based Morphology

We know from Lecture 22, slide 3 that

$$\begin{aligned}
\partial_t u &= \Delta u \\
&= \partial_{xx} u + \partial_{yy} u \quad \text{(by the definition of } \Delta\text{)} \\
&= \partial_{\eta\eta} u + \partial_{\xi\xi} u \quad (\eta \parallel \nabla u, \xi \perp \nabla u)
\end{aligned}$$

We want with MCM a smoothing along isophotes, so $\partial_{\eta\eta} u$ vanishes, since this term smoothes across the edge.
So it remains to show that

$$\begin{aligned}
\partial_{\xi\xi} u &= \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2} && (1)\\
&= \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top \mathrm{Hess}(u) \nabla u && (2)\\
&= |\nabla u| \mathrm{div}\left(\frac{\nabla u}{|\nabla u|}\right) && (3)
\end{aligned}$$

Let's show that (1) = (2):

$$\begin{aligned}
& \Delta u - \frac{1}{|\nabla u|^2} \nabla u^\top \mathrm{Hess}(u) \nabla u \\
= \ & u_{xx} + u_{yy} - \frac{1}{u_x^2 + u_y^2} \nabla u^\top \mathrm{Hess}(u) \nabla u \\
= \ & u_{xx} + u_{yy} - \frac{1}{u_x^2 + u_y^2} (u_x, u_y) \begin{pmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{pmatrix} \begin{pmatrix} u_x \\ u_y \end{pmatrix} \\
= \ & \frac{(u_x^2 + u_y^2)(u_{xx} + u_{yy})}{u_x^2 + u_y^2} - \frac{1}{u_x^2 + u_y^2} (u_x, u_y) \begin{pmatrix} u_{xx} u_x + u_{xy} u_y \\ u_{yx} u_x + u_{yy} u_y \end{pmatrix} \\
= \ & \frac{(u_x^2 + u_y^2)(u_{xx} + u_{yy})}{u_x^2 + u_y^2} - \frac{1}{u_x^2 + u_y^2} \left(u_{xx} u_x^2 + u_{xy} u_y u_x + u_{yx} u_x u_y + u_{yy} u_y^2 \right) \\
= \ & \frac{u_x^2 u_{xx} + u_x^2 u_{yy} + u_y^2 u_{xx} + u_y^2 u_{yy} - u_{xx} u_x^2 - 2u_x u_y u_{xy} - u_{yy} u_y^2}{u_x^2 + u_y^2} \\
= \ & \frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2 + u_y^2}
\end{aligned}$$

Let's show that (1) = (3):

$$
\begin{aligned}
&|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\
= &\sqrt{u_x^2+u_y^2}\left(\partial_x \frac{u_x}{\sqrt{u_x^2+u_y^2}} + \partial_y \frac{u_y}{\sqrt{u_x^2+u_y^2}}\right) \\
= &\sqrt{u_x^2+u_y^2}\left(\frac{u_{xx}\sqrt{u_x^2+u_y^2} - u_x \frac{1}{2\sqrt{u_x^2+u_y^2}}(2u_x u_{xx} + 2u_y u_{yx})}{u_x^2+u_y^2}\right. \\
&\left. + \frac{u_{yy}\sqrt{u_x^2+u_y^2} - u_y \frac{1}{2\sqrt{u_x^2+u_y^2}}(2u_x u_{yx} + 2u_y u_{yy})}{u_x^2+u_y^2}\right) \\
= &\frac{u_{xx}(u_x^2+u_y^2) - u_x(u_x u_{xx} + u_y u_{yx})}{u_x^2+u_y^2} + \frac{u_{yy}(u_x^2+u_y^2) - u_y(u_x u_{yx} + u_y u_{yy})}{u_x^2+u_y^2} \\
= &\frac{u_{xx}u_x^2 + u_{xx}u_y^2 - u_x^2 u_{xx} - u_x u_y u_{xy} + u_{yy}u_x^2 + u_{yy}u_y^2 - u_x u_y u_{xy} - u_y^2 u_{yy}}{u_x^2+u_y^2} \\
= &\frac{u_y^2 u_{xx} - 2u_x u_y u_{xy} + u_x^2 u_{yy}}{u_x^2+u_y^2}
\end{aligned}
$$

6.4 Multiple Choice

a. NO

b. NO

c. NO

d. NO