Differential Equations in Image Processing and Computer Vision

Assignment T4

Jan Hendrik Dithmar
2031259
Group 2: Thu, 12-14 (Markus Mainberger)

1	2	3	4	$\sum$

4.1 Continuous Variational Regularisation

We have
$$E_f(u) = \int_\Omega \left(\Psi_1\left((u-f)^2\right) + \alpha\Psi_2\left(|\nabla u|^2\right)\right) d\mathbf{x}$$
with
$$\begin{aligned}\Psi_1(s^2) &= \sqrt{s^2+\varepsilon^2}\\ \Psi_2(s^2) &= \lambda^2 \log\left(1+s^2/\lambda^2\right)\end{aligned}$$
and $\alpha, \varepsilon, \lambda > 0$.
It follows:
$$E_f(u) = \int_\Omega \left(\sqrt{(u-f)^2+\varepsilon^2} + \alpha\lambda^2\log\left(1+|\nabla u|^2/\lambda^2\right)\right) d\mathbf{x}$$
We have
$$E_f(u) = \int_\Omega F(x_1, x_2, u, u_{x_1}, u_{x_2}) d\mathbf{x}$$
which has to satisfy the Euler-Lagrange equation
$$F_u - \partial_{x_1} F_{u_{x_1}} - \partial_{x_2} F_{u_{x_2}} = 0$$
with the boundary condition
$$\mathbf{n}^\top \begin{pmatrix} F_{u_{x_1}} \\ F_{u_{x_2}} \end{pmatrix} = 0.$$

$$\begin{aligned}
F &= \sqrt{(u-f)^2+\varepsilon^2} + \alpha\lambda^2\log\left(1+|\nabla u|^2/\lambda^2\right)\\
F_u &= \frac{1}{2}\cdot\left((u-f)^2+\varepsilon^2\right)^{-\frac{1}{2}}\cdot 2(u-f)\\
&= \frac{u-f}{\sqrt{(u-f)^2+\varepsilon^2}}\\
F_{u_{x_1}} &= \alpha\lambda^2\frac{1}{1+|\nabla u|^2/\lambda^2}2u_{x_1}/\lambda^2\\
&= 2\alpha\frac{u_{x_1}}{1+|\nabla u|^2/\lambda^2}\\
F_{u_{x_2}} &= \alpha\lambda^2\frac{1}{1+|\nabla u|^2/\lambda^2}2u_{x_2}/\lambda^2\\
&= 2\alpha\frac{u_{x_2}}{1+|\nabla u|^2/\lambda^2}
\end{aligned}$$

Just plugging in the results into the Euler-Lagrange equation yields

$$\begin{aligned}
\frac{u-f}{\sqrt{(u-f)^2+\varepsilon^2}} - \partial_{x_1} 2\alpha \frac{u_{x_1}}{1+|\nabla u|^2/\lambda^2} - \partial_{x_2} 2\alpha \frac{u_{x_2}}{1+|\nabla u|^2/\lambda^2} &= 0 \\
\frac{u-f}{\sqrt{(u-f)^2+\varepsilon^2}} - 2\alpha \partial_{x_1} \frac{u_{x_1}}{1+|\nabla u|^2/\lambda^2} - 2\alpha \partial_{x_2} \frac{u_{x_2}}{1+|\nabla u|^2/\lambda^2} &= 0 \\
\frac{u-f}{\sqrt{(u-f)^2+\varepsilon^2}} - 2\alpha \ \mathrm{div} \left(\frac{\nabla u}{1+|\nabla u|^2/\lambda^2} \right) &= 0
\end{aligned}$$

with the boundary condition

$$\mathbf{n}^\top \begin{pmatrix} F_{u_{x_1}} \\ F_{u_{x_2}} \end{pmatrix} = 0.$$

This knowledge allows to formulate the gradient descent:

$$\begin{aligned}
\partial_t u &= -\gamma \nabla_u E \\
&= -\gamma \cdot \left(\frac{u-f}{\sqrt{(u-f)^2+\varepsilon^2}} - 2\alpha \ \mathrm{div} \left(\frac{\nabla u}{1+|\nabla u|^2/\lambda^2} \right) \right)
\end{aligned}$$

4.2 Rotation Invariance of Energy Functionals

In both cases it should hold the following:

$$\Psi(u_x, u_y) = \Psi(u_\xi, u_\eta)$$

a. $\Psi(u_x, u_y) = |u_x| + |u_y|$.
It follows that $\Psi(u_\xi, u_\eta) = |u_\xi| + |u_\eta|$. From the hint we can get the following for u_ξ and u_η:

$$\begin{aligned}
u_\xi &= u_x \cdot \cos\vartheta + u_y \cdot \sin\vartheta \\
u_\eta &= -u_x \cdot \sin\vartheta + u_y \cdot \cos\vartheta
\end{aligned}$$

$$\Rightarrow \Psi(u_\xi, u_\eta) = |u_x \cdot \cos\vartheta + u_y \cdot \sin\vartheta| + |-u_x \cdot \sin\vartheta + u_y \cdot \cos\vartheta|$$

Let's have a look at the case that $\vartheta = \frac{\pi}{4}$. Then we get

$$\Psi(u_\xi, u_\eta) = \frac{\sqrt{2}}{2}(|u_x + u_y| + |u_y - u_x|) \neq |u_x| + |u_y| = \Psi(u_x, u_y)$$

Thus $\Psi(u_x, u_y) = |u_x| + |u_y|$ is not rotaionally invariant.

b. $\Psi(u_x, u_y) = u_x^2 + u_y^2$.
It follows $\Psi(u_\xi, u_\eta) = u_\xi^2 + u_\eta^2$. Using again the hint, we get for u_ξ and u_η the same as above.

$$\begin{aligned}
\Rightarrow \Psi(u_\xi, u_\eta) &= (u_x \cdot \cos\vartheta + u_y \cdot \sin\vartheta)^2 + (u_y \cdot \cos\vartheta - u_x \sin\vartheta)^2 \\
&= u_x^2 \cos^2\vartheta + 2u_x u_y \cos\vartheta \sin\vartheta + u_y^2 \sin^2\vartheta \\
&\quad + u_y^2 \cos^2\vartheta - 2u_x u_y \cos\vartheta \sin\vartheta + u_x^2 \sin^2\vartheta \\
&= u_x^2 (\underbrace{\cos^2\vartheta + \sin^2\vartheta}_{=1}) + u_y^2 (\underbrace{\cos^2\vartheta + \sin^2\vartheta}_{=1}) \\
&= u_x^2 + u_y^2 = \Psi(u_x, u_y)
\end{aligned}$$

Thus $\Psi(u_x, u_y) = u_x^2 + u_y^2$ is rotationally invariant.

4.3 Discrete Energy Minimisation

$$E_f(u_1, ..., u_N) = \sum_{k=1}^{N} (u_k - f_k)^2 + \alpha \sum_{k=1}^{N-1} (u_{k+1} - u_k)^2 + \beta \sum_{k=2}^{N-1} (u_{k+1} - 2u_k + u_{k-1})^2$$

a. Setting $\frac{\partial E}{\partial u_i} = 0$ for $i = 1, ..., N$ gives

$$\begin{aligned}
0 &= 2(u_1 - f_1) - 2\alpha(u_2 - u_1) + 2\beta(u_3 - 2u_2 + u_1) \\
&= u_1 - f_1 - \alpha(u_2 - u_1) + \beta(u_3 - 2u_2 + u_1)
\end{aligned}$$

$$\begin{aligned}
0 &= 2(u_2 - f_2) - 2\alpha(u_3 - u_2) + 2\alpha(u_2 - u_1) + 2\beta(u_4 - 2u_3 + u_2) \\
&\quad -4\beta(u_3 - 2u_2 + u_1) \\
&= u_2 - f_2 - \alpha(u_3 - u_2) + \alpha(u_2 - u_1) + \beta(u_4 - 2u_3 + u_2) \\
&\quad -2\beta(u_3 - 2u_2 + u_1) \\
0 &= 2(u_i - f_i) - 2\alpha(u_{i+1} - u_i) + 2\alpha(u_i - u_{i-1}) \\
&\quad +2\beta(u_{i+2} - 2u_{i+1} + u_i) - 4\beta(u_{i+1} - 2u_i + u_{i-1}) \\
&\quad +2\beta(u_i - 2u_{i-1} + u_{i-2}) \qquad (i = 3, ..., N-2) \\
&= (u_i - f_i) - \alpha(u_{i+1} - u_i) + \alpha(u_i - u_{i-1}) \\
&\quad +\beta(u_{i+2} - 2u_{i+1} + u_i) - 2\beta(u_{i+1} - 2u_i + u_{i-1}) \\
&\quad +\beta(u_i - 2u_{i-1} + u_{i-2}) \qquad (i = 3, ..., N-2)
\end{aligned}$$

$$\begin{aligned} 0 &= 2(u_{N-1} - f_{N-1}) - 2\alpha(u_N - u_{N-1}) + 2\alpha(u_{N-1} - u_{N-2}) \\ &\quad -4\beta(u_N - 2u_{N-1} + u_{N-2}) + 2\beta(u_{N-1} - 2u_{N-2} + u_{N-3}) \\ &= u_{N-1} - f_{N-1} - \alpha(u_N - u_{N-1}) + \alpha(u_{N-1} - u_{N-2}) \\ &\quad -2\beta(u_N - 2u_{N-1} + u_{N-2}) + \beta(u_{N-1} - 2u_{N-2} + u_{N-3}) \end{aligned}$$

$$\begin{aligned} 0 &= 2(u_N - f_N) + 2\alpha(u_N - u_{N-1}) - 2\beta(u_N - 2u_{N-1} + u_{N-2}) \\ &= u_N - f_N + \alpha(u_N - u_{N-1}) - \beta(u_N - 2u_{N-1} + u_{N-2}) \end{aligned}$$

b. I would say that the terms where a β stands in front of are the discretisations of the fourth order derivatives. If you combine these terms to one term you get a structure where $u_{i+2}, u_{i+1}, u_i, u_{i-1}, u_{i-2}$ are involved, so this is not a second order but a fourth order derivative.

4.4 Half-Quadratic Regularisation

$$E_f(u) = \int_\Omega \left((u-f)^2 + \alpha 2\lambda^2 \left(1 - \exp\left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \right) \right) d\mathbf{x}$$

I can rewrite this into

$$E_f(u) = \int_\Omega \left((u-f)^2 + \alpha \Psi\left(|\nabla u|^2 \right) \right) d\mathbf{x}$$

with

$$\Psi(|\nabla u|^2) = 2\lambda^2 \left(1 - \exp\left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \right)$$

$$\begin{aligned} F &= (u-f)^2 + \alpha\Psi\left(|\nabla u|^2\right) \\ F_u &= 2(u-f) \\ F_{u_{x_1}} &= 2\alpha\lambda^2 \left(-\exp\left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \right) \cdot \left(-2u_{x_1}/2\lambda^2 \right) \\ &= 2\alpha u_{x_1} \exp\left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \\ F_{u_{x_2}} &= 2\alpha\lambda^2 \left(-\exp\left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \right) \cdot \left(-2u_{x_2}/2\lambda^2 \right) \\ &= 2\alpha u_{x_2} \exp\left(-\frac{|\nabla u|^2}{2\lambda^2} \right) \end{aligned}$$

Its Euler-Lagrange equation is given by (divided by 2)

$$0 = (u - f) - \alpha \cdot \ \text{div}\left(\underbrace{\exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right)}_{\Psi'(|\nabla u|^2)} \nabla u\right)$$

Let's formulate the energy functional for the Half-Quadratic case:

$$E_{HQ}(u, v) := \int_\Omega \left((u - f)^2 + \alpha(v \cdot |\nabla u|^2 + \eta(v))\right) d\mathbf{x}$$

$E_{HQ}(u, v)$ has the two Euler-Lagrange equations:

$$\begin{aligned} 0 &= (u - f) - \alpha \ \text{div}(v\nabla u) \\ 0 &= |\nabla u|^2 + \eta'(v) \end{aligned}$$

We know that

$$v := \Psi'(|\nabla u|^2) = \exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right)$$

To obtain $|\nabla u|^2$, let's solve v for $|\nabla u|^2$:

$$\begin{aligned} v &= \exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right) \\ \log(v) &= -\frac{|\nabla u|^2}{2\lambda^2} \\ |\nabla u|^2 &= -\log(v) \cdot 2\lambda^2 \\ &= -2\lambda^2 \cdot \log(v) \end{aligned}$$

Then we can rewrite the Euler-Lagrange equations

$$\begin{aligned} 0 &= (u - f) - \alpha \ \text{div}(v\nabla u) \\ 0 &= |\nabla u|^2 \underbrace{-2\lambda^2 \log(v)}_{=\eta'(v)} \end{aligned}$$

We know that $\eta'(v) = -2\lambda^2 \log(v)$ and thus

$$\eta(v) = -2\lambda^2 v(\log(v) - 1)$$

This gives

$$E_{HQ}(u, v) := \int_\Omega \left((u - f)^2 + \alpha(v|\nabla u|^2 - 2\lambda^2 v(\log(v) - 1))\right) d\mathbf{x}$$