Differential Equations in Image Processing and Computer Vision

Assignment T2

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Group 2: Thu, 12-14 (Markus Mainberger)

2.1 Isotropic Nonlinear Diffusion: Diffusivities

a.

$$
\begin{aligned}
g(s^2) &= \frac{1}{1+\frac{s^2}{\lambda^2}} \\
\Phi(s) &= s \cdot g(s^2) \\
&= s \cdot \frac{1}{1+\frac{s^2}{\lambda^2}} = \frac{s}{1+\frac{s^2}{\lambda^2}} \\
\Phi'(s) &= \frac{1 \cdot \left(1+\frac{s^2}{\lambda^2}\right) - s \cdot \frac{1}{\lambda^2} \cdot 2s}{\left(1+\frac{s^2}{\lambda^2}\right)^2} \\
&= \frac{1}{1+\frac{s^2}{\lambda^2}} - \frac{2s^2}{\left(1+\frac{s^2}{\lambda^2}\right)^2 \cdot \lambda^2} \\
&= \frac{\left(1+\frac{s^2}{\lambda^2}\right) \cdot \lambda^2 - 2s^2}{\left(1+\frac{s^2}{\lambda^2}\right)^2 \cdot \lambda^2} \\
\Phi'(s) &= \frac{\lambda^2 - s^2}{\left(1+\frac{s^2}{\lambda^2}\right)^2 \cdot \lambda^2}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \Phi'(s) > 0 \quad &\Leftrightarrow \quad |s| < \lambda \quad \text{(forward diffusion)} \\
\Phi'(s) < 0 \quad &\Leftrightarrow \quad |s| > \lambda \quad \text{(backward diffusion)}
\end{aligned}
$$

b.

$$
\begin{aligned}
g(s^2) &= \frac{1}{\sqrt{1+\frac{s^2}{\lambda^2}}} \\
\Phi(s) &= \frac{s}{\sqrt{1+\frac{s^2}{\lambda^2}}} \\
\Phi'(s) &= \frac{\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{2}} - s\cdot\frac{1}{2}\cdot\left(1+\frac{s^2}{\lambda^2}\right)\cdot\frac{1}{\lambda^2}\cdot 2s}{1+\frac{s^2}{\lambda^2}} \\
&= \frac{\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{2}} - \frac{s^2}{\lambda^2}\cdot\left(1+\frac{s^2}{\lambda^2}\right)}{1+\frac{s^2}{\lambda^2}} \\
&= \frac{1}{\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{2}} - \frac{s^2}{\lambda^2}} \\
\Phi'(s) &= \frac{\lambda^2 - s^2\cdot\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{2}}}{\lambda^2\cdot\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{2}}}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \Phi'(s) > 0 \quad &\Leftrightarrow \quad |s| < \frac{\lambda}{\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{4}}} \quad \text{(forward diffusion)} \\
\Phi'(s) < 0 \quad &\Leftrightarrow \quad |s| > \frac{\lambda}{\left(1+\frac{s^2}{\lambda^2}\right)^{\frac{1}{4}}} \quad \text{(backward diffusion)}
\end{aligned}
$$

c.

$$
\begin{aligned}
g(s^2) &= \exp\left(\frac{-s^2}{2\lambda^2}\right) \\
\Phi(s) &= s \cdot \exp\left(\frac{-s^2}{2\lambda^2}\right) \\
\Phi'(s) &= \exp\left(-\frac{s^2}{2\lambda^2}\right) + s \cdot \exp\left(-\frac{s^2}{2\lambda^2}\right) \cdot \left(-\frac{2s}{2\lambda^2}\right) \\
&= \exp\left(-\frac{s^2}{2\lambda^2}\right) - \frac{s^2}{\lambda^2} \cdot \exp\left(-\frac{s^2}{2\lambda^2}\right) \\
\Phi'(s) &= \left(1 - \frac{s^2}{\lambda^2}\right) \cdot \underbrace{\exp\left(-\frac{s^2}{2\lambda^2}\right)}_{>0}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \Phi'(s) > 0 \quad &\Leftrightarrow \quad |s| < \lambda \quad \text{(forward diffusion)} \\
\Phi'(s) < 0 \quad &\Leftrightarrow \quad |s| > \lambda \quad \text{(backward diffusion)}
\end{aligned}
$$

2.2 Isotropic Nonlinear Diffusion: Stencil Notation of Explicit Scheme

$$\begin{aligned}\partial_t u &= \operatorname{div}\left(g(|\nabla u|^2)\nabla u\right)\\ &= \partial_x\left(g(|\nabla u|^2)\partial_x u\right) + \partial_y\left(g(|\nabla u|^2)\partial_y u\right)\end{aligned}$$

We know from the lecture (Lecture 5, slide 3) that:

$$\begin{aligned}\frac{du_{i,j}}{dt} &= \frac{1}{h_1}\left(\frac{g_{i+1,j}+g_{i,j}}{2}\frac{u_{i+1,j}-u_{i,j}}{h_1} - \frac{g_{i,j}+g_{i-1,j}}{2}\frac{u_{i,j}-u_{i-1,j}}{h_1}\right)\\ &+\frac{1}{h_2}\left(\frac{g_{i,j+1}+g_{i,j}}{2}\frac{u_{i,j+1}-u_{i,j}}{h_2} - \frac{g_{i,j}+g_{i,j-1}}{2}\frac{u_{i,j}-u_{i,j-1}}{h_2}\right)\end{aligned}$$

Furthermore we know that

$$\begin{aligned}\frac{du}{dt} &= A(u)u\\ \Leftrightarrow \frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\tau} &= \frac{1}{2h_1^2}\left((g_{i+1,j}+g_{i,j})(u_{i+1,j}-u_{i,j}) - (g_{i,j}+g_{i-1,j})(u_{i,j}-u_{i-1,j})\right)\\ &+\frac{1}{2h_2^2}\left((g_{i,j+1}+g_{i,j})(u_{i,j+1}-u_{i,j}) - (g_{i,j}+g_{i,j-1})(u_{i,j}-u_{i,j-1})\right)\\ \Leftrightarrow u_{i,j}^{k+1} &= \frac{\tau}{2h_1^2}(g_{i+1,j}u_{i+1,j} - g_{i+1,j}u_{i,j} + g_{i,j}u_{i+1,j} - g_{i,j}u_{i,j})\\ &-\frac{\tau}{2h_1^2}(g_{i,j}u_{i,j} - g_{i,j}u_{i-1,j} + g_{i-1,j}u_{i,j} - g_{i-1,j}u_{i-1,j})\\ &+\frac{\tau}{2h_2^2}(g_{i,j+1}u_{i,j+1} - g_{i,j+1}u_{i,j} + g_{i,j}u_{i,j+1} - g_{i,j}u_{i,j})\\ &-\frac{\tau}{2h_2^2}(g_{i,j}u_{i,j} - g_{i,j}u_{i,j-1} + g_{i,j-1}u_{i,j} - g_{i,j-1}u_{i,j-1})\\ &+u_{i,j}\\ &= u_{i,j}\left(1 - \tau\frac{g_{i+1,j}+2g_{i,j}+g_{i-1,j}}{2h_1^2} - \tau\frac{g_{i,j+1}+2g_{i,j}+g_{i,j-1}}{2h_2^2}\right)\\ &+u_{i+1,j}\left(\tau\frac{g_{i+1,j}+g_{i,j}}{2h_1^2}\right) + u_{i-1,j}\left(\tau\frac{g_{i,j}+g_{i-1,j}}{2h_1^2}\right)\\ &+u_{i,j+1}\left(\tau\frac{g_{i,j+1}+g_{i,j}}{2h_2^2}\right) + u_{i,j-1}\left(\tau\frac{g_{i,j}+g_{i,j-1}}{2h_2^2}\right)\end{aligned}$$

With this knowledge, we can give the stencil notation:

0	$\tau\frac{g_{i,j+1}+g_{i,j}}{2h_2^2}$	0
$\tau\frac{g_{i,j}+g_{i-1,j}}{2h_1^2}$	$1-\tau\left(\frac{g_{i+1,j}+2g_{i,j}+g_{i-1,j}}{2h_1^2}+\frac{g_{i,j+1}+2g_{i,j}+g_{i,j-1}}{2h_2^2}\right)$	$\tau\frac{g_{i+1,j}+g_{i,j}}{2h_1^2}$
0	$\tau\frac{g_{i,j}+g_{i,j-1}}{2h_2^2}$	0

2.3 Isotropic Nonlinear Diffusion: Matrix Notation of Semi-Implicit Scheme

For a 1-D signal of length 4 we get a 4×4 matrix A which has the following entries:

$$\begin{aligned}
a_{11} &= -\left(\frac{g_1+g_0}{2h^2}+\frac{g_1+g_2}{2h^2}\right) = -\frac{g_0+2g_1+g_2}{2h^2}\\
a_{22} &= -\left(\frac{g_2+g_1}{2h^2}+\frac{g_2+g_3}{2h^2}\right) = -\frac{g_1+2g_2+g_3}{2h^2}\\
a_{33} &= -\left(\frac{g_3+g_2}{2h^2}+\frac{g_3+g_4}{2h^2}\right) = -\frac{g_2+2g_3+g_4}{2h^2}\\
a_{44} &= -\left(\frac{g_4+g_3}{2h^2}+\frac{g_4+g_5}{2h^2}\right) = -\frac{g_3+2g_4+g_5}{2h^2}
\end{aligned}$$

$$\begin{aligned}
a_{12} &= \frac{g_1+g_2}{2h^2}\\
a_{13} &= 0\\
a_{14} &= 0
\end{aligned}$$

$$\begin{aligned}
a_{21} &= \frac{g_1+g_2}{2h^2}\\
a_{23} &= \frac{g_2+g_3}{2h^2}\\
a_{24} &= 0
\end{aligned}$$

$$\begin{aligned}
a_{31} &= 0\\
a_{32} &= \frac{g_2+g_3}{2h^2}\\
a_{34} &= \frac{g_3+g_4}{2h^2}
\end{aligned}$$

$$\begin{aligned}
a_{41} &= 0\\
a_{42} &= 0\\
a_{43} &= \frac{g_3+g_4}{2h^2}
\end{aligned}$$

In the case of a 2-D image of size 3×4, we get a 12×12 matrix A with the following entries (where all the entries not stated here are 0):

$$
\begin{aligned}
a_{1,1} &= -\left(\frac{g_1+g_1}{2h_1^2}+\frac{g_1+g_2}{2h_1^2}+\frac{g_1+g_1}{2h_2^2}+\frac{g_1+g_4}{2h_2^2}\right) \\
&= -\frac{3g_1+g_2}{2h_1^2}-\frac{3g_1+g_4}{2h_2^2} \\
a_{2,2} &= -\left(\frac{g_2+g_1}{2h_1^2}+\frac{g_2+g_3}{2h_1^2}+\frac{g_2+g_2}{2h_2^2}+\frac{g_2+g_5}{2h_2^2}\right) \\
&= -\frac{g_1+2g_2+g_3}{2h_1^2}-\frac{3g_2+g_5}{2h_2^2} \\
a_{3,3} &= -\frac{g_2+3g_3}{2h_1^2}-\frac{3g_3+g_6}{2h_2^2} \\
a_{4,4} &= -\frac{3g_4+g_5}{2h_1^2}-\frac{g_1+2g_4+g_7}{2h_2^2} \\
a_{5,5} &= -\frac{g_4+2g_5+g_6}{2h_1^2}-\frac{g_2+2g_5+g_8}{2h_2^2} \\
a_{6,6} &= -\frac{g_5+3g_6}{2h_1^2}-\frac{g_3+2g_6+g_9}{2h_2^2} \\
a_{7,7} &= -\frac{3g_7+g_8}{2h_1^2}-\frac{g_4+2g_7+g_{10}}{2h_2^2} \\
a_{8,8} &= -\frac{g_7+2g_8+g_9}{2h_1^2}-\frac{g_5+2g_8+g_{11}}{2h_2^2} \\
a_{9,9} &= -\frac{g_8+3g_9}{2h_1^2}-\frac{g_6+2g_9+g_{12}}{2h_2^2} \\
a_{10,10} &= -\frac{3g_{10}+g_{11}}{2h_1^2}-\frac{g_7+3g_{10}}{2h_2^2} \\
a_{11,11} &= -\frac{g_{10}+2g_{11}+g_{12}}{2h_1^2}-\frac{g_8+3g_{11}}{2h_2^2} \\
a_{12,12} &= -\frac{g_{11}+3g_{12}}{2h_1^2}-\frac{g_9+3g_{12}}{2h_2^2}
\end{aligned}
$$

$$a_{1,2} = \frac{g_1 + g_2}{2h_1^2} \qquad a_{1,4} = \frac{g_1 + g_4}{2h_2^2} \qquad a_{2,1} = \frac{g_1 + g_2}{2h_1^2}$$

$$a_{2,3} = \frac{g_2 + g_3}{2h_1^2} \qquad a_{2,5} = \frac{g_2 + g_5}{2h_2^2} \qquad a_{3,2} = \frac{g_2 + g_3}{2h_1^2}$$

$$a_{3,6} = \frac{g_3 + g_6}{2h_2^2} \qquad a_{4,1} = \frac{g_1 + g_4}{2h_2^2} \qquad a_{4,5} = \frac{g_4 + g_5}{2h_1^2}$$

$$a_{4,7} = \frac{g_4 + g_7}{2h_2^2} \qquad a_{5,2} = \frac{g_2 + g_5}{2h_2^2} \qquad a_{5,4} = \frac{g_4 + g_5}{2h_1^2}$$

$$a_{5,6} = \frac{g_5 + g_6}{2h_1^2} \qquad a_{5,8} = \frac{g_5 + g_8}{2h_2^2} \qquad a_{6,3} = \frac{g_3 + g_6}{2h_2^2}$$

$$a_{6,5} = \frac{g_5 + g_6}{2h_1^2} \qquad a_{6,9} = \frac{g_6 + g_9}{2h_2^2} \qquad a_{7,4} = \frac{g_4 + g_7}{2h_2^2}$$

$$a_{7,8} = \frac{g_7 + g_8}{2h_1^2} \qquad a_{7,10} = \frac{g_7 + g_{10}}{2h_2^2} \qquad a_{8,5} = \frac{g_5 + g_8}{2h_2^2}$$

$$a_{8,7} = \frac{g_7 + g_8}{2h_1^2} \qquad a_{8,9} = \frac{g_8 + g_9}{2h_1^2} \qquad a_{8,11} = \frac{g_8 + g_{11}}{2h_2^2}$$

$$a_{9,6} = \frac{g_6 + g_9}{2h_2^2} \qquad a_{9,8} = \frac{g_8 + g_9}{2h_1^2} \qquad a_{9,12} = \frac{g_9 + g_{12}}{2h_2^2}$$

$$a_{10,7} = \frac{g_7 + g_{10}}{2h_2^2} \qquad g_{10,11} = \frac{g_{10} + g_{11}}{2h_1^2} \qquad g_{11,8} = \frac{g_8 + g_{11}}{2h_2^2}$$

$$g_{11,10} = \frac{g_{10} + g_{11}}{2h_1^2} \qquad g_{11,12} = \frac{g_{11} + g_{12}}{2h_1^2} \qquad g_{12,9} = \frac{g_9 + g_{12}}{2h_2^2}$$

$$g_{12,11} = \frac{g_{11} + g_{12}}{2h_1^2}$$

2.4 Linear Systems of Equations and Thomas' Algorithm

a. We want to decompose the matrix

$$A = \begin{pmatrix} \gamma_1 & \delta_1 & \varepsilon_1 & 0 & 0 \\ \beta_1 & \gamma_2 & \delta_2 & \varepsilon_2 & 0 \\ \alpha_1 & \beta_2 & \gamma_3 & \delta_3 & \varepsilon_3 \\ 0 & \alpha_2 & \beta_3 & \gamma_4 & \delta_4 \\ 0 & 0 & \alpha_3 & \beta_4 & \gamma_5 \end{pmatrix}$$

into

$$L := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 0 \\ a_1 & b_2 & 1 & 0 & 0 \\ 0 & a_2 & b_3 & 1 & 0 \\ 0 & 0 & a_3 & b_4 & 1 \end{pmatrix}$$

and

$$R := \begin{pmatrix} c_1 & d_1 & e_1 & 0 & 0 \\ 0 & c_2 & d_2 & e_2 & 0 \\ 0 & 0 & c_3 & d_3 & e_3 \\ 0 & 0 & 0 & c_4 & d_4 \\ 0 & 0 & 0 & 0 & c_5 \end{pmatrix}.$$

The matrix A can be written as follows:

$$A = LR$$

$$= \begin{pmatrix} c_1 & d_1 & e_1 & 0 & 0 \\ b_1c_1 & b_1d_1 + c_2 & b_1e_1 + d_2 & e_2 & 0 \\ a_1c_1 & a_1d_1 + b_2c_2 & a_1e_1 + b_2d_2 + c_3 & b_2e_2 + d_3 & e_3 \\ 0 & a_2c_2 & a_2d_2 + b_3c_3 & a_2e_2 + b_3d_3 + c_4 & b_3e_3 + d_4 \\ 0 & 0 & a_3c_3 & a_3d_3 + b_4c_4 & a_3e_3 + b_4d_4 + c_5 \end{pmatrix}$$

LR-Decomposition:
We know that $e_i = \varepsilon_i, i \in \{1, 2, 3\}$. The coefficients c_i, b_i and a_i can be computed by:

c_1 := γ_1
b_1 := β_1 / c_1
a_1 := α_1 / c_1
d_1 := δ_1

```
for i = 2, ..., 5:
    if i ≤ 4:
      d_i := δ_i - b_{i-1}e_{i-1}

    c_i := γ_i - b_{i-1}d_{i-1} - a_{i-2}e_{i-2}

    if i ≤ 4:
      b_i := β_i / c_i - a_{i-1}d_{i-1}

    if i ≤ 3:
      a_i := α_i / c_i
```

b. *Forward Elimination:*
We solve $L\mathbf{y} = \mathbf{u}$ for $\mathbf{y}$:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 0 \\ a_1 & b_2 & 1 & 0 & 0 \\ 0 & a_2 & b_3 & 1 & 0 \\ 0 & 0 & a_3 & b_4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{pmatrix}$$

```
y_1 := u_1
y_2 := u_2 - b_1y_1
for i = 3, .., 5:
    y_i := u_i - a_{i-2}y_{i-2} - b_{i-1}y_{i-1}
```

Backward Substitution:
We solve $R\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$:

$$\begin{pmatrix} c_1 & d_1 & e_1 & 0 & 0 \\ 0 & c_2 & d_2 & e_2 & 0 \\ 0 & 0 & c_3 & d_3 & e_3 \\ 0 & 0 & 0 & c_4 & d_4 \\ 0 & 0 & 0 & 0 & c_5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix}$$

```
x_5 := y_5 / c_5
x_4 := (y_4 - d_4x_5) / c_4
for i = 3, ..., 1:
    x_i := (y_i - d_ix_{i+1} - e_ix_{i+2}) / c_i
```