Differential Equations in Image Processing and Computer Vision

Assignment T1

Jan Hendrik Dithmar
2031259
Group 2: Thu, 12-14 (Markus Mainberger)

1.1 Convolution, Derivatives and Diffusion Equation

a.

$$
\begin{aligned}
(f * g')(x) &= \int_{\mathbb{R}^2} f(y)(g(x-y))_x dy \qquad (1)\\
(f * g)'(x) &= \int_{\mathbb{R}^2} (f(y)g(x-y))_x dy\\
&= \int_{\mathbb{R}^2} \big[\underbrace{(f(y))_x}_{=0} g(x-y) + f(y)(g(x-y))_x\big] dy\\
(f * g)'(x) &= \int_{\mathbb{R}^2} f(y)(g(x-y))_x dy = (f * g')(x) \qquad (2)\\
(f * g)'(x) &\overset{Comm.}{=} (g * f)'(x)\\
&\overset{(2)}{=} (g * f')(x)\\
&\overset{Comm.}{=} (f' * g)(x)
\end{aligned}
$$

□

b.

$$
\begin{aligned}
u(x,t) &\overset{!}{=} K_{\sqrt{2t}}(x)\\
u_t &= u_{xx} \quad x \in \mathbb{R}, t > 0
\end{aligned}
$$

We know from the lecture that

$$
\begin{aligned}
K_\sigma(x) &= \frac{1}{\sigma\sqrt{2\pi}} \cdot \exp\left(-\frac{x^2}{2\sigma^2}\right)\\
\Rightarrow K_{\sqrt{2t}}(x) &= \frac{1}{2\sqrt{\pi t}} \cdot \exp\left(-\frac{x^2}{4t}\right)
\end{aligned}
$$

I calculate the derivatives u_t and u_{xx} and check whether they are equal.

$$
\begin{aligned}
u_t &= \frac{-\frac{1}{2}\pi}{2(\pi t)^{3/2}} \exp\left(-\frac{x^2}{4t}\right) + \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\frac{x^2}{4t^2}\\
&= -\frac{\pi}{4(\pi t)^{3/2}} \exp\left(-\frac{x^2}{4t}\right) + \frac{x^2}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\\
&= -\frac{\pi}{4\sqrt{\pi t}\pi t} \exp\left(-\frac{x^2}{4t}\right) + \frac{x^2}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\\
&= -\frac{2t}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) + \frac{x^2}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\\
u_t &= \frac{x^2 - 2t}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)
\end{aligned}
$$

$$
\begin{aligned}
u_x &= \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\frac{-2x}{4t} = -\frac{x}{4t\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) \\
u_{xx} &= -\frac{1}{4t\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) - \frac{x}{4t\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)\left(-\frac{x}{2t}\right) \\
&= -\frac{1}{4t\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) - \frac{x^2}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) \\
&= -\frac{2t}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) + \frac{x^2}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) \\
u_{xx} &= \frac{x^2-2t}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)
\end{aligned}
$$

One can easily see that $u_t = u_{xx}$ which means that $u(x,t) = K_{\sqrt{2t}}(x)$ is a solution. □

c. In this case $u(x,t) = (K_{\sqrt{2t}} * f)(x)$. One can easily show that $u(x, t = 0) = f(x)$ by using the given assumption:

$$
u(x, t=0) = (K_0 * f)(x) \overset{Ass.}{=} f(x) \quad \checkmark
$$

Now, one only has to show that $u_t = u_{xx}$.

$$
\begin{aligned}
u_t &= ((K_{\sqrt{2t}} * f)(x))_t \\
\overset{(a)}{\Leftrightarrow} u_t &= ((K_{\sqrt{2t}})_t * f)(x)
\end{aligned}
$$

$$
\begin{aligned}
u_{xx} &= ((K_{\sqrt{2t}} * f)(x))_{xx} \\
\overset{(a)}{\Leftrightarrow} u_{xx} &= ((K_{\sqrt{2t}})_{xx} * f)(x)
\end{aligned}
$$

Since we know from part **b.**, that

$$
\begin{aligned}
(K_{\sqrt{2t}})_t &= \frac{x^2-2t}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) \\
(K_{\sqrt{2t}})_x &= -\frac{x}{4t\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right) \\
(K_{\sqrt{2t}})_{xx} &= \frac{x^2-2t}{8t^2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)
\end{aligned}
$$

we see that $(K_{\sqrt{2t}})_t$ and $(K_{\sqrt{2t}})_{xx}$ are the same.
Since $(K_{\sqrt{2t}})_t$ and $(K_{\sqrt{2t}})_{xx}$ are the same, the terms $((K_{\sqrt{2t}})_t * f)(x) = ((K_{\sqrt{2t}} * f)(x))_t$ and $((K_{\sqrt{2t}})_{xx} * f)(x) = ((K_{\sqrt{2t}} * f)(x))_{xx}$ are the same for any given bounded continuous funtion f. □

1.2 Numerical Differentiation

We want to approximate f'' with $f_{i-2}, f_{i-1}, f_i, f_{i+1}, f_{i+2}$. Doing a Taylor expansion results in the following:

$$\begin{aligned}
f_{i-2} &= f_i - 2hf_i' + 2h^2 f_i'' - \frac{4}{3}h^3 f_i''' + \frac{2}{3}h^4 f_i^{(4)} - \frac{4}{15}h^5 f_i^{(5)} + O(h^6) \\
f_{i-1} &= f_i - hf_i' + \frac{1}{2}h^2 f_i'' - \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} - \frac{1}{120}h^5 f_i^{(5)} + O(h^6) \\
f_i &= f_i \\
f_{i+1} &= f_i + hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} + \frac{1}{120}h^5 f_i^{(5)} + O(h^6) \\
f_{i+2} &= f_i + 2hf_i' + 2h^2 f_i'' + \frac{4}{3}h^3 f_i''' + \frac{2}{3}h^4 f_i^{(4)} + \frac{4}{15}h^5 f_i^{(5)} + O(h^6)
\end{aligned}$$

Comparing the coefficients:

$$0\cdot f_i + 0\cdot f_i' + 1\cdot f_i'' + 0\cdot f_i''' + 0\cdot f_i^{(4)} \overset{!}{=} \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2}$$

$$\begin{aligned}
f_i'' = {} & (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2) f_i \\
& + (-2h\alpha_{-2} - h\alpha_{-1} + h\alpha_1 + 2h\alpha_2) f_i' \\
& + \left(2h^2\alpha_{-2} + \frac{1}{2}h^2\alpha_{-1} + \frac{1}{2}h^2\alpha_1 + 2h^2\alpha_2\right) f_i'' \\
& + \left(-\frac{4}{3}h^3\alpha_{-2} - \frac{1}{6}h^3\alpha_{-1} + \frac{1}{6}h^3\alpha_1 + \frac{4}{3}h^3\alpha_2\right) f_i''' \\
& + \left(\frac{2}{3}h^4\alpha_{-2} + \frac{1}{24}h^4\alpha_{-1} + \frac{1}{24}h^4\alpha_1 + \frac{2}{3}h^4\alpha_2\right) f_i^{(4)} + O(h^5)
\end{aligned}$$

This gives the following system of equations:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 2h^2 & \frac{1}{2}h^2 & 0 & \frac{1}{2}h^2 & 2h^2 \\ -\frac{4}{3}h^3 & -\frac{1}{6}h^3 & 0 & \frac{1}{6}h^3 & \frac{4}{3}h^3 \\ \frac{2}{3}h^4 & \frac{1}{24}h^4 & 0 & \frac{1}{24}h^4 & \frac{2}{3}h^4 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2/h^2 \\ 0 \\ 0 \end{pmatrix}$$

Solving the equation system: (checked with Maple, everything's fine *gg*)

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 4 & 1 & 0 & 1 & 4 & 2/h^2 \\ -8 & -1 & 0 & 1 & 8 & 0 \\ 16 & 1 & 0 & 1 & 16 & 0 \end{array}\right) \quad +2\cdot IV$$

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 4 & 1 & 0 & 1 & 4 & 2/h^2 \\ -8 & -1 & 0 & 1 & 8 & 0 \\ 0 & -1 & 0 & 3 & 32 & 0 \end{array}\right) \quad -4\cdot II$$

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 4 & 1 & 0 & 1 & 4 & 2/h^2 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & -1 & 0 & 3 & 32 & 0 \end{array}\right) \quad +\tfrac{1}{3}\cdot IV; (:2)$$

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 4 & 1 & 0 & 1 & 4 & 2/h^2 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 & 0 \end{array}\right) \quad +2\cdot II$$

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 0 & -1 & 0 & 3 & 8 & 2/h^2 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 16 & 0 \end{array}\right) \quad +\tfrac{1}{3}\cdot IV; (:2)$$

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ -2 & -1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1/h^2 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 16 & 0 \end{array}\right) \quad +2\cdot I$$

$$\left(\begin{array}{rrrrr|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1/h^2 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 16 & 0 \end{array}\right) \quad -V$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & -12 & -1/h^2 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 16 & 0 \end{array}\right) \begin{array}{r} \Rightarrow \alpha_{-2} = -\frac{1}{12h^2} \\ \Rightarrow \alpha_0 = -\frac{5}{2h^2} \\ \Rightarrow \alpha_2 = -\frac{1}{12h^2} \\ \Rightarrow \alpha_{-1} = \frac{4}{3h^2} \\ \Rightarrow \alpha_1 = \frac{4}{3h^2} \end{array}$$

Finally we get:

$$\begin{aligned} \alpha_{-2} = \alpha_2 &= -\frac{1}{12h^2} \\ \alpha_{-1} = \alpha_1 &= \frac{4}{3h^2} \\ \alpha_0 &= -\frac{5}{2h^2} \end{aligned}$$

Let's plug that into

$$\begin{aligned} f_i'' &= \underbrace{(\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2)}_{=0} f_i \\ &+ \underbrace{(-2h\alpha_{-2} - h\alpha_{-1} + h\alpha_1 + 2h\alpha_2)}_{=0} f_i' \\ &+ \underbrace{\left(2h^2\alpha_{-2} + \frac{1}{2}h^2\alpha_{-1} + \frac{1}{2}h^2\alpha_1 + 2h^2\alpha_2\right)}_{=1} f_i'' \\ &+ \underbrace{\left(-\frac{4}{3}h^3\alpha_{-2} - \frac{1}{6}h^3\alpha_{-1} + \frac{1}{6}h^3\alpha_1 + \frac{4}{3}h^3\alpha_2\right)}_{=0} f_i''' \\ &+ \underbrace{\left(\frac{2}{3}h^4\alpha_{-2} + \frac{1}{24}h^4\alpha_{-1} + \frac{1}{24}h^4\alpha_1 + \frac{2}{3}h^4\alpha_2\right)}_{=0} f_i^{(4)} \\ &+ \underbrace{\left(-\frac{4}{15}h^5\alpha_{-2} - \frac{1}{120}h^5\alpha_{-1} + \frac{1}{120}h^5\alpha_1 + \frac{4}{15}h^5\alpha_2\right)}_{=0} f_i^{(5)} \\ &+ \underbrace{\left(\frac{4}{45}h^6\alpha_{-2} + \frac{1}{720}h^6\alpha_{-1} + \frac{1}{720}h^6\alpha_1 + \frac{4}{45}h^6\alpha_2\right)}_{=-\frac{1}{90}h^4} f_i^{(6)} + O(h^7) \\ f_i'' &= f_i'' - \frac{1}{90}h^4 f_i^{(6)} + O(h^7) \end{aligned}$$

It follows that the order of consistency is 4.