Differential Equations in Image Processing and Computer Vision 2008
Example Solutions for Theoretical Assignments 6 (T6)

Problem 1

We deal with discretising $u_t = au_x$.

(a) We focus our attention on the spatial discretisation, i.e., in all considered variations we use

$$u_t \approx \frac{u_i^{n+1} - u_i^n}{\tau},$$

where the spatial index i corresponds to the point ih, and where n and $n+1$ denote the time level $n\tau$ and $(n+1)\tau$, respectively.

Backward differences:

$$u_i^{n+1} = u_i^n + a\frac{\tau}{h}\left(u_i^n - u_{i-1}^n\right)$$

Forward differences:

$$u_i^{n+1} = u_i^n + a\frac{\tau}{h}\left(u_{i+1}^n - u_i^n\right)$$

Central differences:

$$u_i^{n+1} = u_i^n + a\frac{\tau}{h}\left(u_{i+1}^n - u_{i-1}^n\right)$$

(b) **Backward differences:** We rewrite the formula in terms of a convex combination, i.e.,

$$u_i^{n+1} = u_i^n + a\frac{\tau}{h}\left(u_i^n - u_{i-1}^n\right) = \left(1 + a\frac{\tau}{h}\right)u_i^n - a\frac{\tau}{h}u_{i-1}^n\,.$$

The formula only makes sense if $a < 0$ and if $1 + |a|\,\frac{\tau}{h} \geq 0$. Concerning the latter condition, we see that the scheme is only stable for

$$0 < \tau \leq \frac{h}{|a|}\,.$$

Forward differences: Also here, we rewrite the formula in terms of a convex combination, i.e.,

$$u_i^{n+1} = u_i^n + a\frac{\tau}{h}\left(u_{i+1}^n - u_i^n\right) = \left(1 - a\frac{\tau}{h}\right)u_i^n + a\frac{\tau}{h}u_{i+1}^n\,.$$

The formula only makes sense if $a > 0$ and if $1 - a\frac{\tau}{h} \geq 0$. Concerning the latter condition, we see that the scheme is only stable for

$$0 < \tau \leq \frac{h}{a}\,.$$

Central differences: It is not possible to rewrite the scheme in terms of a convex combination. Moreover, we see by the non-uniform (i.e., both positive and negative) weighting influence in

$$u_i^{n+1} = u_i^n + a\frac{\tau}{h}\left(u_{i+1}^n - u_{i-1}^n\right) = u_i^n + a\frac{\tau}{h}u_{i+1}^n - a\frac{\tau}{h}u_{i-1}^n\,,$$

that we can construct situations violating a minimum-maximum principle: for instance, taking data

$$u_i^0 = \left\{\begin{array}{ccc} 0 & : & i \leq 0 \\ 1 & : & i \geq 1 \end{array}\right. ,$$

together with $a < 0$, we easily see that u_0^1 is always negative, whereas for $a > 0$, u_1^1 is always larger than 1.

Problem 2 (Slope Transform)

We calculate the slope transform:

$$\begin{aligned}
\mathcal{S}\left[f_p\right](u) &= \operatorname{stat}_x\left(f_p(x) - ux\right) \\
&= \left\{f_p(x) - ux | f_p'(x) - u = 0\right\} \\
&= \left\{cx^p - ux | pcx^{p-1} - u = 0\right\} \\
&= \left\{x\left(cx^{p-1} - u\right) | x^{p-1} = \frac{u}{pc}\right\} \\
&= \left\{x\left(\frac{u}{p} - u\right) | x = \operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}}\right\} \\
&= \left\{\operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}}\frac{u}{p}(1-p)\right\} \\
\mathcal{S}\left[f_2\right](u) &= \left\{-\frac{u^2}{4c}\right\}
\end{aligned}$$

So, for the case $p = 2$, the slope transform remains a quadratic function, i.e. paraboloids are preserved under the slope transform.

Problem 3 (Curvature-Based Morphology)

Let $\xi = (\xi_1, \xi_2)^\top$ be a unit vector. Then the second derivative of u in direction of ξ is given by

$$\begin{aligned}
\partial_{\xi\xi}u &= \begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix}^\top \begin{pmatrix}u_{xx} & u_{xy} \\ u_{xy} & u_{yy}\end{pmatrix}\begin{pmatrix}\xi_1 \\ \xi_2\end{pmatrix} \\
&= \xi_1^2 u_{xx} + 2\xi_1\xi_2 u_{xy} + \xi_2^2 u_{yy}\,.
\end{aligned}$$

Knowing that ξ is a unit vector that is orthogonal to the flow gradient $\nabla u = (u_x, u_y)^\top$ we obtain

$$\xi = \frac{1}{|\nabla u|}\begin{pmatrix}-u_y \\ u_x\end{pmatrix}$$

so that

$$\begin{aligned}
\xi_1 &= \frac{-u_y}{|\nabla u|}\,, \\
\xi_2 &= \frac{u_x}{|\nabla u|}\,.
\end{aligned}$$

Consequently

$$\begin{aligned}
\partial_{\xi\xi} u &= \xi_1^2 u_{xx} + 2\xi_1\xi_2 u_{xy} + \xi_2^2 u_{yy} \\
&= \frac{u_y^2 u_{xx}}{|\nabla u|^2} - 2\frac{u_x u_y u_{xy}}{|\nabla u|^2} + \frac{u_x^2 u_{yy}}{|\nabla u|^2} \\
&= \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{u_x^2 + u_y^2}\,.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\partial_{\xi\xi} u &= \Delta u - \frac{1}{|\nabla u|^2}\nabla u^\top \mathrm{Hess}(u)\nabla u \\
&= u_{xx} + u_{yy} - \frac{1}{u_x^2+u_y^2}\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top \begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix} \\
&= (u_{xx}+u_{yy})\underbrace{\frac{\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top \begin{pmatrix}u_x\\u_y\end{pmatrix}}{u_x^2+u_y^2}}_{1} - \frac{1}{u_x^2+u_y^2}\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top \begin{pmatrix}u_{xx} & u_{xy}\\ u_{xy} & u_{yy}\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix} \\
&= \frac{1}{u_x^2+u_y^2}\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top (u_{xx}+u_{yy})\, I \begin{pmatrix}u_x\\u_y\end{pmatrix} + \frac{1}{u_x^2+u_y^2}\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top \begin{pmatrix}-u_{xx} & -u_{xy}\\ -u_{xy} & -u_{yy}\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix} \\
&= \frac{1}{u_x^2+u_y^2}\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top \begin{pmatrix}u_{xx}+u_{yy}-u_{xx} & -u_{xy}\\ -u_{xy} & u_{xx}+u_{yy}-u_{yy}\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix} \\
&= \frac{1}{u_x^2+u_y^2}\begin{pmatrix}u_x\\u_y\end{pmatrix}^\top \begin{pmatrix}u_{yy} & -u_{xy}\\ -u_{xy} & u_{xx}\end{pmatrix}\begin{pmatrix}u_x\\u_y\end{pmatrix} \\
&= \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{u_x^2+u_y^2}\,.
\end{aligned}$$

Finally we have

$$
\begin{aligned}
\partial_{\xi\xi} u &= |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \\
&= |\nabla u| \frac{\partial}{\partial x}\left(\frac{u_x}{|\nabla u|}\right) + |\nabla u| \frac{\partial}{\partial y}\left(\frac{u_y}{|\nabla u|}\right) \\
&= |\nabla u| \left(u_{xx}\frac{1}{|\nabla u|} - u_x \frac{1}{2|\nabla u|^3}(2u_x u_{xx} + 2u_y u_{xy})\right) \\
&\quad + |\nabla u| \left(u_{yy}\frac{1}{|\nabla u|} - u_y \frac{1}{2|\nabla u|^3}(2u_x u_{xy} + 2u_y u_{yy})\right) \\
&= u_{xx} - \frac{u_x(u_x u_{xx} + u_y u_{xy})}{|\nabla u|^2} + u_{yy} - \frac{u_y(u_x u_{xy} + u_y u_{yy})}{|\nabla u|^2} \\
&= \frac{(u_{xx} + u_{yy})(u_x^2 + u_y^2) - u_x(u_x u_{xx} + u_y u_{xy}) - u_y(u_x u_{xy} + u_y u_{yy})}{u_x^2 + u_y^2} \\
&= \frac{u_x^2 u_{yy} - 2u_x u_y u_{xy} + u_y^2 u_{xx}}{u_x^2 + u_y^2}\,.
\end{aligned}
$$

Problem 4 (Multiple Choice)

The following questions had to be answered with YES or NO

- Let us consider the corresponding semi-implicit scheme from Lecture 19, Slide 5. The extension of this scheme for 3-D optic flow computation requires the solution of three penta-diagonal systems of equations each step.
  NO. Although the extension to a third dimension will give us three equation systems, we obtain hepta-diagonal systems of equations instead of penta-diagonal ones. This is due to the fact that in 3-D we have seven neighbours in the isotropic case instead of five.

- Gaussians are invariant under the Slope transform.
  NO. Gaussians are invariant under the Fourier transform. Functions that are invariant under the Slope transform are paraboloids.

- Mean curvature motion preserves the average grey value.
  NO. Unlike the diffusion processes discussed before in the lecture, mean curvature motion can not be written in divergence form. An effect of this fact is that mean curvature motion does in general not preserve the average grey value.

- The shape inclusion principle for affine morphological scale-space only holds for ellipses.
  NO. Although the AMSS has many properties related to ellipses, the shape inclusion principle holds for all forms.