Differential Equations in Image Processing and Computer Vision 2008
**Example Solutions for Theoretical Assignments 5 (T5)**

## Problem 1 (Eigenvalues and Eigenvectors of Nagel's Method)

(a) The calculation

$$(\mathbf{v}\mathbf{v}^\top)\,\mathbf{v} \;=\; \mathbf{v}\,(\mathbf{v}^\top\mathbf{v}) \;=\; \langle \mathbf{v},\mathbf{v}\rangle\cdot\mathbf{v} \;=\; \|\mathbf{v}\|^2\cdot\mathbf{v}$$

shows that $\mathbf{v}$ is an eigenvector of the matrix $\mathbf{v}\mathbf{v}^\top$ with corresponding eigenvalue $\|\mathbf{v}\|^2$. For each vector $\mathbf{w}$ orthogonal to $\mathbf{v}$ we have

$$(\mathbf{v}\mathbf{v}^\top)\,\mathbf{w} \;=\; \mathbf{v}\,(\mathbf{v}^\top\mathbf{w}) \;=\; \langle \mathbf{v},\mathbf{w}\rangle\cdot\mathbf{v} \;=\; 0$$

Since the vectors $\mathbf{w}$ orthogonal to $\mathbf{v}$ constitute the kernel of the matrix, an $n-1$-dimensional space. Thus we have found the complete representation of the matrix $\mathbf{v}\mathbf{v}^\top$ in eigenvalues and eigenvectors. Since the kernel has dimension $n-1$, the rank of the matrix is 1.

(b) Given are

$$\begin{aligned}
D(\nabla f) &= \frac{1}{|\nabla f|^2+2\lambda^2}\left(\nabla f^\perp\left(\nabla f^\perp\right)^\top+\lambda^2\mathcal{I}\right),\\
v_1 &= \nabla f\,,\\
v_2 &= \nabla f^\perp\,,\\
\lambda_1 &= \frac{\lambda^2}{|\nabla f|^2+2\lambda^2}\,,\\
\lambda_2 &= \frac{|\nabla f|^2+\lambda^2}{|\nabla f|^2+2\lambda^2}\,.
\end{aligned}$$

We now verify that the equalities

$$\begin{aligned}
D(\nabla f)\,v_1 &= \lambda_1 v_1\,, &(1)\\
D(\nabla f)\,v_2 &= \lambda_2 v_2 &(2)
\end{aligned}$$

hold.

<u>To (1):</u>

$$\begin{aligned}
&D(\nabla f)\,v_1\\
&= \frac{1}{|\nabla f|^2+2\lambda^2}\left(\nabla f^\perp\left(\nabla f^\perp\right)^\top+\lambda^2\mathcal{I}\right)\nabla f\\
&= \frac{1}{|\nabla f|^2+2\lambda^2}\nabla f^\perp\underbrace{\left(\nabla f^\perp\right)^\top\nabla f}_{=0}+\frac{1}{|\nabla f|^2+2\lambda^2}\lambda^2\underbrace{\mathcal{I}\nabla f}_{=\nabla f}\\
&= \frac{\lambda^2}{|\nabla f|^2+2\lambda^2}\nabla f \quad=\quad \lambda_1 v_1\,.
\end{aligned}$$

To (2):

$$
\begin{aligned}
& D\,(\nabla f)\, v_2 \\
= \quad & \frac{1}{|\nabla f|^2 + 2\lambda^2} \left( \nabla f^\perp \left( \nabla f^\perp \right)^\top + \lambda^2 \mathcal{I} \right) \nabla f^\perp \\
= \quad & \frac{1}{|\nabla f|^2 + 2\lambda^2} \nabla f^\perp \underbrace{\left( \nabla f^\perp \right)^\top \nabla f^\perp}_{=|\nabla f|^2} + \frac{1}{|\nabla f|^2 + 2\lambda^2} \lambda^2 \underbrace{\mathcal{I} \nabla f^\perp}_{=\nabla f^\perp} \\
= \quad & \frac{|\nabla f|^2}{|\nabla f|^2 + 2\lambda^2} \nabla f^\perp + \frac{\lambda^2}{|\nabla f|^2 + 2\lambda^2} \nabla f^\perp \\
= \quad & \frac{|\nabla f|^2 + \lambda^2}{|\nabla f|^2 + 2\lambda^2} \nabla f^\perp \quad = \quad \lambda_2 v_2\,.
\end{aligned}
$$

**Problem 2 (Motion Tensors)**

The task is to compute the motion tensors for the following penalisers:

(a) constancy of the Laplacian,

(b) constancy of the spatial image gradient magnitude of an RGB image.

(a) **Constancy of the Laplacian:**
There is one constancy assumption, namely

$$u^\top \nabla_3 \Delta_2 f \;=\; 0 \;, \quad p \;:=\; p_1 \;=\; \Delta_2 f \;=\; f_{x_1x_1} + f_{x_2x_2} \;\;.$$

Thus the penaliser reads

$$M \;=\; \left(u^\top \nabla_3 p\right)^2 \;,$$

and the motion tensor is given by

$$J\left(\nabla_3 p\right) \;=\; \nabla_3 p \nabla_3 p^\top \;.$$

The computation of $\nabla_3 p$ yields

$$\begin{aligned}
\nabla_3 p \;&=\; \nabla_3(f_{x_1x_1} + f_{x_2x_2}) \;=\; \begin{pmatrix} \dfrac{\partial}{\partial x_1}(f_{x_1x_1} + f_{x_2x_2}) \\ \dfrac{\partial}{\partial x_2}(f_{x_1x_1} + f_{x_2x_2}) \\ \dfrac{\partial}{\partial x_3}(f_{x_1x_1} + f_{x_2x_2}) \end{pmatrix} \\
&=\; \begin{pmatrix} f_{x_1x_1x_1} + f_{x_1x_2x_2} \\ f_{x_1x_1x_2} + f_{x_2x_2x_2} \\ f_{x_1x_1x_3} + f_{x_2x_2x_3} \end{pmatrix} \;.
\end{aligned}$$

Then the complete motion tensor reads as

$$\begin{aligned}
&\nabla_3 p \nabla_3 p^\top \\
&\quad=\; \begin{pmatrix} f_{x_1x_1x_1} + f_{x_1x_2x_2} \\ f_{x_1x_1x_2} + f_{x_2x_2x_2} \\ f_{x_1x_1x_3} + f_{x_2x_2x_3} \end{pmatrix} \cdot \begin{pmatrix} f_{x_1x_1x_1} + f_{x_1x_2x_2} \\ f_{x_1x_1x_2} + f_{x_2x_2x_2} \\ f_{x_1x_1x_3} + f_{x_2x_2x_3} \end{pmatrix}^\top \\
&\quad=\; \Big((f_{x_ix_1x_1} + f_{x_ix_2x_2})(f_{x_jx_1x_1} + f_{x_jx_2x_2})\Big)_{i,j=1,2,3} \;\;.
\end{aligned}$$

(b) **Constancy of the spatial image gradient magnitude of an RGB image:**
The given RGB image sequence has the three channels $f_1, f_2,$and $f_3$. As it has been described in Lecture 14 in the context of regularisation and diffusion filtering, we use the sum of the squared gradient magnitudes of the three channels

$$\begin{aligned}
p \;&=\; \sqrt{|\nabla_2 f_1|^2 + |\nabla_2 f_2|^2 + |\nabla_2 f_3|^2} \\
&=\; \sqrt{f_{1,x_1}^2 + f_{1,x_2}^2 + f_{2,x_1}^2 + f_{2,x_2}^2 + f_{3,x_1}^2 + f_{3,x_2}^2}
\end{aligned}$$

as a feature for our constancy assumption. We calculate

$$\nabla_3 p = \begin{pmatrix} \frac{\partial}{\partial x_1} p \\ \frac{\partial}{\partial x_2} p \\ \frac{\partial}{\partial x_3} p \end{pmatrix} = \frac{1}{p} \begin{pmatrix} \sum_{k=1}^{3} f_{k,x_1} f_{k,x_1 x_1} + f_{k,x_2} f_{k,x_1 x_2} \\ \sum_{k=1}^{3} f_{k,x_1} f_{k,x_1 x_2} + f_{k,x_2} f_{k,x_2 x_2} \\ \sum_{k=1}^{3} f_{k,x_1} f_{k,x_1 x_3} + f_{k,x_2} f_{k,x_2 x_3} \end{pmatrix}$$

The corresponding motion tensor then consists of the entries

$$\begin{aligned} &\nabla_3 p \nabla_3 p^\top \\ &= \frac{1}{p^2} \left( \Big( \sum_{k=1}^{3} f_{k,x_1} f_{k,x_1 x_i} + f_{k,x_2} f_{k,x_2 x_i} \Big) \Big( \sum_{k=1}^{3} f_{k,x_1} f_{k,x_1 x_j} + f_{k,x_2} f_{k,x_2 x_j} \Big) \right)_{i,j=1,2,3} . \end{aligned}$$

**Problem 3 (Least Squares in Motion Estimation)**

(a) Given the data term

$$M = \sum_{i=1}^{n} \gamma_i \, \left(\mathbf{u}^\top \, \nabla_3 \, p_i\right)^2 = \mathbf{u}^\top J \, \mathbf{u}$$

that expresses $n$ constancy assumptions on the image features $p_1, ..., p_n$ weighted with the scalars $\lambda_1, ..., \lambda_n$, we can derive the following least squares problem:

$$\begin{aligned}
&\sum_{i=1}^{n} \gamma_i \, (\mathbf{u}^\top \, \nabla_3 \, p_i)^2 \\
&= \quad \sum_{i=1}^{n} \gamma_i \, \left((u_1, u_2, 1) \, (p_{i_{x_1}}, p_{i_{x_2}}, p_{i_{x_3}})^\top\right)^2 \\
&= \quad \sum_{i=1}^{n} \left((u_1, u_2) \, (\sqrt{\gamma_i} \, p_{i_{x_1}}, \sqrt{\gamma_i} \, p_{i_{x_2}})^\top - (-\sqrt{\gamma_i} \, p_{i_{x_3}})\right)^2 \\
&= \quad \underbrace{\left| \begin{pmatrix} \sqrt{\gamma_1} \, p_{1_{x_1}} & \sqrt{\gamma_1} \, p_{1_{x_2}} \\ \vdots & \vdots \\ \sqrt{\gamma_n} \, p_{n_{x_1}} & \sqrt{\gamma_n} \, p_{n_{x_2}} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} -\sqrt{\gamma_1} \, p_{1_{x_3}} \\ \vdots \\ -\sqrt{\gamma_n} \, p_{n_{x_3}} \end{pmatrix} \right|^2}_{|A\tilde{\mathbf{u}} - \mathbf{b}|^2} .
\end{aligned}$$

(b) In order to compute the solution of this least squares fit, we have to minimise it. This can be done by setting its partial derivatives with respect to $u_1$ and $u_2$ to zero, i.e. by setting

$$\frac{\partial}{\partial u_1} \sum_{i=1}^{n} \gamma_i \, (\mathbf{u}^\top \, \nabla_3 \, p_i)^2 = 0$$

$$\frac{\partial}{\partial u_2} \sum_{i=1}^{n} \gamma_i \, (\mathbf{u}^\top \, \nabla_3 \, p_i)^2 = 0$$

and by solving the resulting $2 \times 2$ system of equations. For our general data term this system is given by

$$\underbrace{\begin{pmatrix} \sum_{i=1}^{n} \gamma_i \, p_{i_{x_1}}^2 & \sum_{i=1}^{n} \gamma_i \, p_{i_{x_1}} p_{i_{x_2}} \\ \sum_{i=1}^{n} \gamma_i \, p_{i_{x_1}} p_{i_{x_2}} & \sum_{i=1}^{n} \gamma_i \, p_{i_{x_2}}^2 \end{pmatrix}}_{A^\top A} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{\tilde{\mathbf{u}}} = \underbrace{\begin{pmatrix} -\sum_{i=1}^{n} \gamma_i \, p_{i_{x_1}} p_{i_{x_3}} \\ -\sum_{i=1}^{n} \gamma_i \, p_{i_{x_2}} p_{i_{x_3}} \end{pmatrix}}_{A^\top \mathbf{b}} .$$

If we denote by $J_{nm}(\nabla_3 \, p_1, .., \nabla_3 \, p_n)$ the entry of the motion tensor at position $(n, m)$ this system can be reformulated as

$$\begin{aligned}
&\begin{pmatrix} J_{11}(\nabla_3 \, p_1, .., \nabla_3 \, p_n) & J_{12}(\nabla_3 \, p_1, .., \nabla_3 \, p_n) \\ J_{12}(\nabla_3 \, p_1, .., \nabla_3 \, p_n) & J_{22}(\nabla_3 \, p_1, .., \nabla_3 \, p_n) \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\
&= \quad \begin{pmatrix} -J_{13}(\nabla_3 \, p_1, .., \nabla_3 \, p_n) \\ -J_{23}(\nabla_3 \, p_1, .., \nabla_3 \, p_n) \end{pmatrix} .
\end{aligned}$$

(c) This $2 \times 2$ linear system of equations can only be solved uniquely, if the matrix $A^\top A$ is invertible, i.e. it has rank 2. However, depending on the contributions of the feature gradients $\nabla p_i$, this matrix can also become locally singular, i.e. we have locally rank 1 or 0. Let us now discuss these three scenarios:

- *Rank 2 – Optimal Rank.* The matrix $A^\top A$ is invertible and the data term offers a locally unique solution. Thereby the fact that both eigenvalues are non-zero indicates that at least two of the image features $p_1, ..., p_n$ vary in different spatial directions. This in turn means that the corresponding spatial gradients $\nabla_2 p_i$ and $\nabla_2 p_j$ of these two features are *linearly independent.* Since each of these constraint allows to recover exactly the flow information parallel to its spatial feature gradient (aperture problem), these two linearly independent constraints are sufficient to determine the two unknowns $u_1$ and $u_2$ uniquely. The corresponding solution is then given by

$$\tilde{\mathbf{u}} = (A^\top A)^{-1} A^\top \mathbf{b}. \tag{3}$$

- *Rank 1 – Rank Deficiency of 1.* The matrix $A^\top A$ is not invertible. Since at least one solution of the underlying least square fit must exist, this means that the data term has infinitely many solutions and is thus locally non-unique. Thereby the fact that only one eigenvalue is non-zero shows that all spatial feature gradients $\nabla_2 p_1, .., \nabla_2 p_n$ are *locally dependent*, but at least one of them gives information i.e. $\exists\, i \in \{1, ., n\} : \nabla_2 p_i \neq 0$. Exactly in direction of this non-zero feature gradient, the component of the optic flow can be computed. It is given by

$$\tilde{\mathbf{u}} = (A^\top A)^{+} A^\top \mathbf{b}, \tag{4}$$

  where $(A^\top A)^+$ is the Moore-Penrose pseudoinverse of $(A^\top A)$. While the component in direction of the non-zero feature gradient can be computed, the component orthogonal to it cannot be determined. This results in an *ambiguity* of the solution of rank-1 spatial motion tensors *along a line.*

  One should note that the classical aperture problem resulting from the brightness constancy assumption that was discussed in the introduction is only a special case of this scenario: If only one constraint is used, it is evident that this constraint cannot be sufficient to determine the two unknowns $u_1$ and $u_2$ uniquely. The component of the optic flow that can be computed in this case – the component in direction of the non-zero brightness gradient – is known as the *normal flow.*

- *Rank 0 – Rank Deficiency of 2.* The matrix $A^\top A$ tensor is not invertible. As in the previous case the data term has infinitely many solutions and is thus locally non-unique. Since both eigenvalues are zero, this means that there are no spatial variations of the image features, i.e. $p_{i_{x_1}} = 0$ and $p_{i_{x_2}} = 0$. As a consequence, all entries of the spatial motion tensor and the right hand side become zero. This in turn means that all values for $u_1$ and $u_2$ are local solutions of the data term. Thus, we have an *ambiguity* of the solution of rank-0 spatial motion tensors *in the whole image plane*.

**Problem 4 (Design of Global Optic Flow Methods)**

In order to design an energy functional that allows to compute the optic flow under varying illumination and rotational motion, we need a constancy assumption on a feature that fulfills these two constraints. Such a feature is for instance the Laplacian. If we assume small displacements, the corresponding data term with quadratic penalisation is given by

$$M(\mathbf{u}) := (\mathbf{u}^\top \nabla_3(\Delta_2 f))^2.$$

The simplest spatio-temporal smoothness term is based on homogeneous regularisation. It is given by

$$S(\nabla \mathbf{u}) := |\nabla_3 u_1|^2 + |\nabla_3 u_2|^2,$$

where $\nabla_3$ is the spatio-temporal gradient. Combining these two terms results in the following energy functional:

$$E(\mathbf{u}) = \int\limits_{\Omega \times T} (\mathbf{u}^\top \nabla_3(\Delta_2 f))^2 + \alpha(|\nabla_3 u_1|^2 + |\nabla_3 u_2|^2) \;\, \mathrm{d}\mathbf{x}\, \mathrm{d}t.$$

Please recall that a spatio-temporal method requires the integration over time and that we have to modify the integration domain accordingly (extend the integration domain by the temporal domain $T$).

The corresponding Euler–Lagrange equations for our new functional are given by

$$\begin{aligned}
0 &= (f_{x_1x_1x_1} + f_{x_2x_2x_1})(f_{x_1x_1x_1} + f_{x_2x_2x_1})u_1 \\
&+ (f_{x_1x_1x_1} + f_{x_2x_2x_1})(f_{x_1x_1x_2} + f_{x_2x_2x_2})u_2 \\
&+ (f_{x_1x_1x_1} + f_{x_2x_2x_1})(f_{x_1x_1x_3} + f_{x_2x_2x_3}) \\
&- \alpha\Delta_3 u_1 \;,
\end{aligned}$$

$$\begin{aligned}
0 &= (f_{x_1x_1x_2} + f_{x_2x_2x_2})(f_{x_1x_1x_1} + f_{x_2x_2x_1})u_1 \\
&+ (f_{x_1x_1x_2} + f_{x_2x_2x_2})(f_{x_1x_1x_2} + f_{x_2x_2x_2})u_2 \\
&+ (f_{x_1x_1x_2} + f_{x_2x_2x_2})(f_{x_1x_1x_3} + f_{x_2x_2x_3}) \\
&- \alpha\Delta_3 u_2 \;,
\end{aligned}$$

where the spatio-temporal Laplacian is defined as $\Delta_3 u := u_{x_1x_1} + u_{x_2x_2} + u_{x_3x_3}$.