Differential Equations in Image Processing and Computer Vision 2008
Example Solutions for Theoretical Assignments 4 (T4)

Problem 1 (Continuous Variational Regularisation)

The minimiser of the 2-D energy functional

$$E_f(u) = \int_\Omega \underbrace{\Big(\Psi_1\big((u-f)^2\big) + \alpha\Psi_2\big(|\nabla u|^2\big)\Big)}_{F(x_1,x_2,u,u_{x_1},u_{x_2})} \,\mathrm{d}x$$

satisfies the Euler-Lagrange equation (see Lecture 10, Pages 17–19)

$$\underbrace{F_u - \frac{\partial}{\partial x_1}F_{u_{x_1}} - \frac{\partial}{\partial x_2}F_{u_{x_2}}}_{\nabla_u E} = 0 ,$$

with natural boundary conditions

$$n^\top \begin{pmatrix} F_{u_{x_1}} \\ F_{u_{x_2}} \end{pmatrix} = 0$$

at the image boundary $\partial\Omega$ with normal vector n.

In our case we have

$$\begin{aligned} F &= \Psi_1\big((u-f)^2\big) + \alpha\Psi_2\big(u_{x_1}^2 + u_{x_2}^2\big) \\ F_u &= 2(u-f)\Psi_1'\big((u-f)^2\big) \\ F_{u_{x_1}} &= 2\alpha\Psi_2'\big(u_{x_1}^2 + u_{x_2}^2\big)u_{x_1} \\ F_{u_{x_2}} &= 2\alpha\Psi_2'\big(u_{x_1}^2 + u_{x_2}^2\big)u_{x_2} \;. \end{aligned}$$

Then the associated Euler-Lagrange equation reads

$$\begin{aligned} 0 &= F_u - \frac{\partial}{\partial x_1}F_{u_{x_1}} - \frac{\partial}{\partial x_2}F_{u_{x_2}} \\ &= 2(u-f)\Psi_1'\big((u-f)^2\big) - 2\alpha\left(\frac{\partial}{\partial x_1}\Big(\Psi_2'\big(u_{x_1}^2+u_{x_2}^2\big)u_{x_1}\Big) \right. \\ &\qquad\qquad \left. + \frac{\partial}{\partial x_2}\Big(\Psi_2'\big(u_{x_1}^2+u_{x_2}^2\big)u_{x_2}\Big)\right) \\ &= (u-f)\Psi_1'\big((u-f)^2\big) - \alpha\,\mathrm{div}\Big(\Psi_2'\big(|\nabla u|^2\big)\nabla u\Big) \;, \end{aligned}$$

with Neumann boundary conditions

$$0 = n^\top \nabla u = \partial_n u \;.$$

The gradient descent (see Lecture 11, Pages 16–17) searches for the steady state ($t \to \infty$) of the evolution equation

$$\partial_t u = -\beta\; \nabla_u E \;,$$

with speed factor $\beta > 0$.

In our case, the corresponding gradient descent reads

$$\partial_t u \;=\; \beta\,\left(\alpha\,\mathrm{div}\Big(\Psi_2'\big(|\nabla u|^2\big)\nabla u\Big) - (u-f)\Psi_1'\big((u-f)^2\big)\right)\;.$$

In particular, from the data term penaliser $\Psi_1(s^2) = \sqrt{s^2+\epsilon^2}$ one obtains the weight function

$$\Psi_1'(s^2) \;=\; \frac{\partial}{\partial s^2}\Psi_1(s^2) \;=\; \frac{1}{2\sqrt{s^2+\epsilon^2}}\;,$$

and from the smoothness term penaliser $\Psi_2(s^2) = \lambda^2 \ln\left(1 + s^2/\lambda^2\right)$ one obtains the diffusivity function

$$\Psi_2'(s^2) \;=\; \frac{\partial}{\partial s^2}\Psi_2(s^2) \;=\; \left(1+\frac{s^2}{\lambda^2}\right)^{-1}\;.$$

Problem 2 (Rotation Invariance of Energy Functionals)

The energy functional

$$E_f(u) \;=\; \int_\Omega \Big((u-f)^2 + \alpha\Psi(u_x,u_y)\Big)\;\mathrm{d}x\;\mathrm{d}y$$

is called rotationally invariant if and only if

$$E_{Rf}(Ru) \;=\; E_f(u) \tag{1}$$

for all images f, u and all rotations R. As already described on the assignment sheet, this is equivalent to the rotational invariance of the penaliser, that means

$$\Psi(u_\xi, u_\eta) \;=\; \Psi(u_x, u_y)$$

where

$$\begin{aligned}\xi &= x\cos\vartheta + y\sin\vartheta\,,\\ \eta &= -x\sin\vartheta + y\cos\vartheta\;.\end{aligned}$$

(a) First we consider the penaliser $\Psi(u_x,u_y) = |u_x| + |u_y|$ which is not rotationally invariant.
A counterexample is given in the following: Fix $\vartheta = \pi/4$, $u_x = -1, u_y = +1$. This choice is possible since for rotational invariance, the condition (1) must hold for all ϑ and all u.) Then $\cos\vartheta = \sin\vartheta = \sqrt{2}/2$, and we have

It follows that

$$|u_\xi| + |u_\eta| \;=\; \left|-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\right| + \left|\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}\right| \;=\; \sqrt{2}\;,$$

on the other hand we have

$$|u_x| + |u_y| \;=\; 2 \;,$$

and so $\Psi(u_x, u_y)$ and the corresponding energy functional are not rotationally invariant.

(b) The second penaliser $\Psi(u_x, u_y) = u_x^2 + u_y^2$ is rotationally invariant. This can be proven by the following calculation:

$$\begin{aligned}
u_\xi^2 + u_\eta^2 &= (u_x \cos\vartheta + u_y \sin\vartheta)^2 + (-u_x \sin\vartheta + u_y \cos\vartheta)^2 \\
&= u_x^2 \cos^2\vartheta + 2u_x u_y \cos\vartheta \sin\vartheta + u_y^2 \sin^2\vartheta \\
&\quad + u_x^2 \sin^2\vartheta - 2u_x u_y \sin\vartheta \cos\vartheta + u_y^2 \cos^2\vartheta \\
&= (u_x^2 + u_y^2)(\cos^2\vartheta + \sin^2\vartheta) \\
&= u_x^2 + u_y^2 \quad \text{for all } \vartheta \;.
\end{aligned}$$

Problem 3 (Discrete Energy Minimisation)

(a) Given is the following discrete energy function:

$$\begin{aligned}
E_f\,(u_1, \ldots, u_N) \;&=\; \sum_{k=1}^{N} (u_k - f_k)^2 + \alpha \sum_{k=1}^{N-1} (u_{k+1} - u_k)^2 \\
&\quad + \beta \sum_{k=2}^{N-1} (u_{k+1} - 2u_k + u_{k-1})^2 \;.
\end{aligned}$$

The *necessary* criterion for obtaining a minimum is

$$0 \overset{!}{=} \left(\frac{\partial E}{\partial u_1}, \ldots, \frac{\partial E}{\partial u_N} \right) \;.$$

This means componentwise:

$$\begin{aligned}
0 = \frac{\partial E}{\partial u_1} &= 2\,(u_1 - f_1) - 2\alpha\,(u_2 - u_1) + 2\beta\,(u_3 - 2u_2 + u_1)\;, \\
0 = \frac{\partial E}{\partial u_2} &= 2\,(u_2 - f_2) + 2\alpha\,(u_2 - u_1) - 2\alpha\,(u_3 - u_2) \\
&\quad + 2\beta\,(u_4 - 2u_3 + u_2) - 4\beta\,(u_3 - 2u_2 + u_1)\;, \\
0 = \frac{\partial E}{\partial u_3} &= 2\,(u_3 - f_3) + 2\alpha\,(u_3 - u_2) - 2\alpha\,(u_4 - u_3) \\
&\quad + 2\beta\,(u_5 - 2u_4 + u_3) - 4\beta\,(u_4 - 2u_3 + u_2) \\
&\quad + 2\beta\,(u_3 - 2u_2 + u_1)\;,
\end{aligned}$$

$$
\begin{aligned}
0 = \frac{\partial E}{\partial u_k} \quad &= \quad 2\,(u_k - f_k) + 2\alpha\,(u_k - u_{k-1}) - 2\alpha\,(u_{k+1} - u_k) \\
&\quad + 2\beta\,(u_{k+2} - 2u_{k+1} + u_k) - 4\beta\,(u_{k+1} - 2u_k + u_{k-1}) \\
&\quad + 2\beta\,(u_k - 2u_{k-1} + u_{k-2}) \\
&\quad \text{for } k = 4, \ldots, N-2, \\
0 = \frac{\partial E}{\partial u_{N-1}} \quad &= \quad 2\,(u_{N-1} - f_{N-1}) + 2\alpha\,(u_{N-1} - u_{N-2}) \\
&\quad - 2\alpha\,(u_N - u_{N-1}) - 4\beta\,(u_N - 2u_{N-1} + u_{N-2}) \\
&\quad + 2\beta\,(u_{N-1} - 2u_{N-2} + u_{N-3})\ , \\
0 = \frac{\partial E}{\partial u_N} \quad &= \quad 2\,(u_N - f_N) - 2\alpha\,(u_N - u_{N-1}) \\
&\quad + 2\beta\,(u_N - 2u_{N-1} + u_{N-2})\ .
\end{aligned}
$$

Therefore, our signal $(u_1^*, \ldots, u_N^*)$ must satisfy the following systems of equations in order to be a minimiser of E_f:

$$
A \begin{pmatrix} u_1^* \\ u_2^* \\ u_3^* \\ \vdots \\ u_{N-2}^* \\ u_{N-1}^* \\ u_N^* \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{N-2} \\ f_{N-1} \\ f_N \end{pmatrix} . \tag{2}
$$

The corresponding system matrix can be found in Table 1.

(b) Let us rewrite the expression for $\partial E / \partial u_k$ identifying especially the terms multiplied by α and β:

$$
\begin{aligned}
\frac{\partial E}{\partial u_k} \quad &= \quad 2\,(u_k - f_k) + 2\alpha\,(u_k - u_{k-1}) - 2\alpha\,(u_{k+1} - u_k) \\
&\quad + 2\beta\,(u_{k+2} - 2u_{k+1} + u_k) - 4\beta\,(u_{k+1} - 2u_k + u_{k-1}) \\
&\quad + 2\beta\,(u_k - 2u_{k-1} + u_{k-2}) \\
&= \quad 2\,(u_k - f_k) - 2\alpha\,(u_{k-1} - 2u_k + u_{k+1}) \\
&\quad + 2\beta\,(u_{k-2} - 4u_{k-1} + 6u_k - 4u_{k+1} + u_{k+2})\ .
\end{aligned}
$$

From this expression we easily see that the second derivative of u is smoothed by α, while β is smoothing of a fourth order derivative, namely $u_{k-2} - 4u_{k-1} + 6u_k - 4u_{k+1} + u_{k+2}$.

$$
\begin{pmatrix}
(1+\alpha+\beta) & -(\alpha+2\beta) & \beta & & & \\
-(\alpha+2\beta) & (1+2\alpha+5\beta) & -(\alpha+4\beta) & \beta & & \\
\beta & -(\alpha+4\beta) & (1+2\alpha+6\beta) & -(\alpha+4\beta) & \beta & \\
\ddots & \ddots & \ddots & \ddots & \ddots & \\
 & \beta & -(\alpha+4\beta) & (1+2\alpha+6\beta) & -(\alpha+4\beta) & \beta \\
 & & \beta & -(\alpha+4\beta) & (1+2\alpha+5\beta) & -(\alpha+2\beta) \\
 & & & \beta & -(\alpha+2\beta) & (1+\alpha+\beta)
\end{pmatrix}
$$

Table 1: System matrix for the linear system of equations (2).

Problem 4 (Half-Quadratic Regularisation)

The minimiser of the 2-D *nonquadratic* functional

$$E_f(u) = \int_\Omega \left((u-f)^2 + \alpha \cdot \underbrace{2\lambda^2 \left(1 - \exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right)\right)}_{\Psi(|\nabla u|^2) \text{ Perona-Malik II penaliser}} \right) \mathrm{d}x$$

satisfies the *nonlinear* PDE (Euler-Lagrange equation)

$$0 = (u-f) - \alpha \operatorname{div}\Big(\Psi'\big(|\nabla u|^2\big)\nabla u\Big) .$$

Such a minimiser should also be the solution of an energy functional that is *quadratic* in u (i.e. creates *linear* problems) and possesses the following structure

$$E_{HQ}(u) = \int_\Omega \Big((u-f)^2 + \alpha \cdot \big(v \cdot |\nabla u|^2 + \eta(v)\big) \Big) \,\mathrm{d}x ,$$

where v is an edge-indicator function and η is a *convex* function in v. (Note that v is positive as derivative of the penaliser.)

The minimiser of E_{HQ} satisfies the following Euler-Lagrange equations:

$$0 = (u-f) - \alpha \operatorname{div}\big(v\nabla u\big) \quad (3)$$

$$0 = |\nabla u|^2 + \eta'(v) . \quad (4)$$

Since E_f and E_{HQ} should have the same minimiser, their Euler-Lagrange equations should hence be equivalent. This requirement is fulfilled by setting the edge-indicator function v in equation (3) as

$$v := \Psi'\big(|\nabla u|^2\big) = \frac{\partial}{\partial |\nabla u|^2} \Psi(|\nabla u|^2) = \exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right) . \quad (5)$$

Equation (4) is obtained by solving v for $|\nabla u|^2$ in equation (5):

$$\begin{aligned} v &= \exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right) \\ \Longleftrightarrow \quad \ln v &= -\frac{|\nabla u|^2}{2\lambda^2} \\ \Longleftrightarrow \quad |\nabla u|^2 &= -2\lambda^2 \ln v \ . \end{aligned}$$

Thus, the convex function η is obtained as

$$\eta(v) = 2\lambda^2 \,(v \ln v - v) .$$

Finally, the Half-Quadratic Regularisation formulation E_{HQ} of E_f is given by

$$E_{HQ}(u,v) = \int_\Omega \Big((u-f)^2 + \alpha \cdot \big(v \cdot |\nabla u|^2 + 2\lambda^2\,(v \ln v - v)\big) \Big) \,\mathrm{d}x ,$$

where $v = \exp\left(-\frac{|\nabla u|^2}{2\lambda^2}\right)$.