Differential Equations in Image Processing and Computer Vision 2008
Example Solutions for Theoretical Assignments 3 (T3)

Problem 1 (Discretisation of Anisotropic Diffusion)

a) Discretisation of the mixed derivative $\partial_y(b\partial_x u)$:

- using forward differences for ∂_y and ∂_x

$$\begin{aligned}\partial_y(b\partial_x u) &\approx \partial_y\left(b_{i,j}\frac{u_{i+1,j}-u_{i,j}}{h_x}\right)\\ &\approx \frac{1}{h_y}\left(b_{i,j+1}\frac{u_{i+1,j+1}-u_{i,j+1}}{h_x}-b_{i,j}\frac{u_{i+1,j}-u_{i,j}}{h_x}\right).\end{aligned}$$

- using backward differences for ∂_y and ∂_x

$$\begin{aligned}\partial_y(b\partial_x u) &\approx \partial_y\left(b_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h_x}\right)\\ &\approx \frac{1}{h_y}\left(b_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h_x}-b_{i,j-1}\frac{u_{i,j-1}-u_{i-1,j-1}}{h_x}\right).\end{aligned}$$

- averaging both results

$$\begin{aligned} & \frac{1}{2h_y}\left(b_{i,j+1}\frac{u_{i+1,j+1}-u_{i,j+1}}{h_x}-b_{i,j}\frac{u_{i+1,j}-u_{i,j}}{h_x}\right)\\ & +\frac{1}{2h_y}\left(b_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h_x}-b_{i,j-1}\frac{u_{i,j-1}-u_{i-1,j-1}}{h_x}\right)\\ = & \frac{1}{2h_yh_x}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i,j+1}\right)-b_{i,j-1}\left(u_{i,j-1}-u_{i-1,j-1}\right)\\ & \quad +b_{i,j}\left(u_{i,j}-u_{i-1,j}\right)-b_{i,j}\left(u_{i+1,j}-u_{i,j}\right)\Big)\\ = & \frac{1}{2h_yh_x}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i,j+1}\right)-b_{i,j-1}\left(u_{i,j-1}-u_{i-1,j-1}\right)\\ & \quad -b_{i,j}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)\Big)\end{aligned}$$

b) Again discretisation of the mixed derivative $\partial_y(b\partial_x u)$:

- using forward differences for ∂_y and backward differences for ∂_x

$$\begin{aligned}\partial_y(b\partial_x u) &\approx \partial_y\left(b_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h_x}\right)\\ &\approx \frac{1}{h_y}\left(b_{i,j+1}\frac{u_{i,j+1}-u_{i-1,j+1}}{h_x}-b_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h_x}\right).\end{aligned}$$

- using backward differences for ∂_y and forward differences for ∂_x

$$\begin{aligned}\partial_y(b\partial_x u) &\approx \partial_y\left(b_{i,j}\frac{u_{i+1,j}-u_{i,j}}{h_x}\right)\\ &\approx \frac{1}{h_y}\left(b_{i,j}\frac{u_{i+1,j}-u_{i,j}}{h_x}-b_{i,j-1}\frac{u_{i+1,j-1}-u_{i,j-1}}{h_x}\right).\end{aligned}$$

- averaging both results

$$\begin{aligned}
&\frac{1}{2h_y}\left(b_{i,j+1}\frac{u_{i,j+1}-u_{i-1,j+1}}{h_x}-b_{i,j}\frac{u_{i,j}-u_{i-1,j}}{h_x}\right)\\
&+\frac{1}{2h_y}\left(b_{i,j}\frac{u_{i+1,j}-u_{i,j}}{h_x}-b_{i,j-1}\frac{u_{i+1,j-1}-u_{i,j-1}}{h_x}\right)\\
=\ &\frac{1}{2h_yh_x}\Big(b_{i,j+1}\left(u_{i,j+1}-u_{i-1,j+1}\right)-b_{i,j-1}\left(u_{i+1,j-1}-u_{i,j-1}\right)\\
&\quad+b_{i,j}\left(u_{i+1,j}-u_{i,j}\right)-b_{i,j}\left(u_{i,j}-u_{i-1,j}\right)\Big)\\
=\ &\frac{1}{2h_yh_x}\Big(b_{i,j+1}\left(u_{i,j+1}-u_{i-1,j+1}\right)-b_{i,j-1}\left(u_{i+1,j-1}-u_{i,j-1}\right)\\
&\quad+b_{i,j}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)\Big)
\end{aligned}$$

What results for the approximation of the entire mixed derivative expression $\partial_y\left(b\partial_x u\right)+\partial_x\left(b\partial_y u\right)$ when the approximations a) and b) are used? (Note: meant are here only the averaged expressions, i.e., the 3rd point under a) and b), respectively.

to a)

The expression for $\partial_x\left(b\partial_y u\right)$ analogous to the *averaged approximation* of $\partial_y\left(b\partial_x u\right)$ reads

$$\begin{aligned}
&\frac{1}{2h_xh_y}\Big(b_{i+1,j}\left(u_{i+1,j+1}-u_{i+1,j}\right)-b_{i-1,j}\left(u_{i-1,j}-u_{i-1,j-1}\right)\\
&\quad-b_{i,j}\left(u_{i,j+1}-2u_{i,j}+u_{i,j-1}\right)\Big)
\end{aligned}$$

Thus, the approximation of $\partial_y\left(b\partial_x u\right)+\partial_x\left(b\partial_y u\right)$ reads

$$\begin{aligned}
&\frac{1}{2h_yh_x}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i,j+1}\right)-b_{i,j-1}\left(u_{i,j-1}-u_{i-1,j-1}\right)\\
&\quad-b_{i,j}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)\Big)\\
+\ &\frac{1}{2h_xh_y}\Big(b_{i+1,j}\left(u_{i+1,j+1}-u_{i+1,j}\right)-b_{i-1,j}\left(u_{i-1,j}-u_{i-1,j-1}\right)\\
&\quad-b_{i,j}\left(u_{i,j+1}-2u_{i,j}+u_{i,j-1}\right)\Big)\\
=\ &\frac{1}{2h_xh_y}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i,j+1}\right)-b_{i,j-1}\left(u_{i,j-1}-u_{i-1,j-1}\right)\\
&\quad+b_{i+1,j}\left(u_{i+1,j+1}-u_{i+1,j}\right)-b_{i-1,j}\left(u_{i-1,j}-u_{i-1,j-1}\right)\\
&\quad-b_{i,j}\left(u_{i,j+1}+u_{i+1,j}-4u_{i,j}+u_{i,j-1}+u_{i-1,j}\right)\Big)
\end{aligned}$$

to b)

The expression for $\partial_x\left(b\partial_y u\right)$ analogous to the *averaged approximation* of $\partial_y\left(b\partial_x u\right)$ reads

$$\begin{aligned}
&\frac{1}{2h_xh_y}\Big(b_{i+1,j}\left(u_{i+1,j}-u_{i+1,j-1}\right)-b_{i-1,j}\left(u_{i-1,j+1}-u_{i-1,j}\right)\\
&\quad+b_{i,j}\left(u_{i,j+1}-2u_{i,j}+u_{i,j-1}\right)\Big)
\end{aligned}$$

Then the approximation of $\partial_y\left(b\partial_x u\right) + \partial_x\left(b\partial_y u\right)$ reads

$$
\begin{aligned}
& \frac{1}{2h_yh_x}\Big(b_{i,j+1}\left(u_{i,j+1}-u_{i-1,j+1}\right)-b_{i,j-1}\left(u_{i+1,j-1}-u_{i,j-1}\right)\\
&\qquad +b_{i,j}\left(u_{i+1,j}-2u_{i,j}+u_{i-1,j}\right)\Big)\\
+\quad & \frac{1}{2h_xh_y}\Big(b_{i+1,j}\left(u_{i+1,j}-u_{i+1,j-1}\right)-b_{i-1,j}\left(u_{i-1,j+1}-u_{i-1,j}\right)\\
&\qquad +b_{i,j}\left(u_{i,j+1}-2u_{i,j}+u_{i,j-1}\right)\Big)\\
=\quad & \frac{1}{2h_xh_y}\Big(b_{i,j+1}\left(u_{i,j+1}-u_{i-1,j+1}\right)-b_{i,j-1}\left(u_{i+1,j-1}-u_{i,j-1}\right)\\
&\qquad\quad +b_{i+1,j}\left(u_{i+1,j}-u_{i+1,j-1}\right)-b_{i-1,j}\left(u_{i-1,j+1}-u_{i-1,j}\right)\\
&\qquad\quad +b_{i,j}\left(u_{i,j+1}+u_{i+1,j}-4u_{i,j}+u_{i,j-1}+u_{i-1,j}\right)\Big)
\end{aligned}
$$

Averaging the last two expressions for the approximation of $\partial_y\left(b\partial_x u\right)+\partial_x\left(b\partial_y u\right)$ yields

$$
\begin{aligned}
& \frac{1}{4h_xh_y}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i,j+1}\right)-b_{i,j-1}\left(u_{i,j-1}-u_{i-1,j-1}\right)\\
&\qquad +b_{i+1,j}\left(u_{i+1,j+1}-u_{i+1,j}\right)-b_{i-1,j}\left(u_{i-1,j}-u_{i-1,j-1}\right)\\
&\qquad -b_{i,j}\left(u_{i,j+1}+u_{i+1,j}-4u_{i,j}+u_{i,j-1}+u_{i-1,j}\right)\Big)\\
+\quad & \frac{1}{4h_xh_y}\Big(b_{i,j+1}\left(u_{i,j+1}-u_{i-1,j+1}\right)-b_{i,j-1}\left(u_{i+1,j-1}-u_{i,j-1}\right)\\
&\qquad +b_{i+1,j}\left(u_{i+1,j}-u_{i+1,j-1}\right)-b_{i-1,j}\left(u_{i-1,j+1}-u_{i-1,j}\right)\\
&\qquad +b_{i,j}\left(u_{i,j+1}+u_{i+1,j}-4u_{i,j}+u_{i,j-1}+u_{i-1,j}\right)\Big)\\
=\quad & \frac{1}{4h_xh_y}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i,j+1}+u_{i,j+1}-u_{i-1,j+1}\right)\\
&\qquad +b_{i+1,j}\left(u_{i+1,j+1}-u_{i+1,j}+u_{i+1,j}-u_{i+1,j-1}\right)\\
&\qquad -b_{i,j-1}\left(u_{i,j-1}-u_{i-1,j-1}+u_{i+1,j-1}-u_{i,j-1}\right)\\
&\qquad -b_{i-1,j}\left(u_{i-1,j}-u_{i-1,j-1}+u_{i-1,j+1}-u_{i-1,j}\right)\Big)\\
=\quad & \frac{1}{4h_xh_y}\Big(b_{i,j+1}\left(u_{i+1,j+1}-u_{i-1,j+1}\right)+b_{i+1,j}\left(u_{i+1,j+1}-u_{i+1,j-1}\right)\\
&\qquad -b_{i,j-1}\left(u_{i+1,j-1}-u_{i-1,j-1}\right)-b_{i-1,j}\left(u_{i-1,j+1}-u_{i-1,j-1}\right)\Big)
\end{aligned}
$$

The final result corresponds to the method introduced in the lecture!

Problem 2 (Discretisation of Anisotropic Nonlinear Diffusion)

In view of the $(i-1,j),(i+1,j)$-components of the 'nonnegativity' stencil, we have to find a **positive definite** matrix

$$
D = \begin{pmatrix} a & b \\ b & c \end{pmatrix}
$$

such that

$$
a - |b| < 0. \tag{1}
$$

Likewise, considering the $(i, j+1), (i, j-1)$-components, one may find a corresponding matrix with

$$c - |b| < 0. \tag{2}$$

A extremely simple example for (1) and (2) would be

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

These matrices are not diagonally dominant. And it becomes apparent that negative entries in the stencil cannot occur if D is diagonally dominant. Under the assumption that the diagonal elements a and c are positive, equations (1) and (2) are the negation of positive definiteness. Note that diagonal dominance implies positive definiteness. The converse is not true, in general, see examples above. However, if for the associated spectral condition number $\kappa < 5.8....$ holds, the converse is true.

Problem 3 (Stopping Time Selection: Decorrelation Criterion)

The signal for which we like to compute the stopping time for linear diffusion with the stencil $(0.25, 0.5.0.25))$ using the decorrelation criterion is given by

$$f = (\ 3,\ 5,\ 3,\ 5, -5, -3, -5, -3)$$

and has mean $\bar{f} = E(f) = 0$.

- **Iteration 1**.

  After the first iteration we obtain the signal

  $$f_1 = (\ 3.500,\ 4.000,\ 4.000,\ 2.000, -2.000, -4.000, -4.000, -3.500)$$

  with mean

  $$\bar{f_1} = E(f_1) = 0$$

  and variance

  $$\text{var}(f_1) = E((f_1 - \bar{f_1})^2) = E(f_1^2) = 12.0625\ .$$

  The corresponding noise is given by

  $$\begin{aligned} n_1 &= f_0 - f_1 \\ &= (-0.500,\ 1.000, -1.000,\ 3.000, -3.000,\ 1.000, -1.000,\ 0.500) \end{aligned}$$

  with mean

  $$\bar{n_1} = E(n_1) = 0$$

  and variance

  $$\text{var}(n_1) = E((n_1 - \bar{n}_1)^2) = E(n_1^2) = 2.8125\ .$$

  The covariance between signal and noise thus reads

  $$\text{cov}(f_1, n_1) = E((f_1 - \bar{f_1})(n_1 - \bar{n}_1)) = E(f_1 n_1) = 1.0625\ .$$

  Consequently the correlation coefficient is given by

  $$\text{corr}(f_1, n_1) = \frac{\text{cov}(f_1, n_1)}{\sqrt{\text{var}(f_1), \text{var}(n_1)}} = \mathbf{0.18242}\ .$$

- **Iteration 2**.

  After the second iteration we obtain the signal

  $$f_2 = (\ 3.625,\ 3.875,\ 3.500,\ 1.500, -1.500, -3.500, -3.875, -3.625)$$

  with mean

  $$\bar{f_2} = E(f_2) = 0$$

  and variance

  $$\mathrm{var}(f_2) = E((f_2 - \bar{f_2})^2) = E(f_2^2) = 10.664\ .$$

  The corresponding noise is given by

  $$\begin{aligned} n_2 &= f_0 - f_2 \\ &= (-0.625,\ 1.125, -0.500,\ 3.500, -3.500,\ 0.500, -1.125,\ 0.625) \end{aligned}$$

  with mean

  $$\bar{n_2} = E(n_2) = 0$$

  and variance

  $$\mathrm{var}(n_2) = E((n_2 - \bar{n}_2)^2) = E(n_2^2) = 3.5390\ .$$

  The covariance between signal and noise thus reads

  $$\mathrm{cov}(f_2, n_2) = E((f_2 - \bar{f_2})(n_2 - \bar{n}_2)) = E(f_2 n_2) = 1.3984\ .$$

  Consequently the correlation coefficient is given by

  $$\mathrm{corr}(f_2, n_2) = \frac{\mathrm{cov}(f_2, n_2)}{\sqrt{\mathrm{var}(f_2), \mathrm{var}(n_2)}} = \mathbf{0.22763}\ .$$

Since the correlation coefficient increases, one should take the result from the *first* iteration. Please recall, however, that there exists no guarantee that this solution constitutes a unique minimum of the correlation.

Problem 4 (Convex Functionals and Forward Diffusion)

We consider the diffusion process

$$\partial_t u \ = \ \partial_x \Big(\Psi'(u_x^2) u_x \Big) \ . \tag{3}$$

To determine the forward-backward diffusion behaviour, we are interested in the flux function $\Phi(u_x) := \Psi'(u_x^2)u_x$. To rewrite (3) as

$$\partial_t u \ = \ \Phi'(u_x) u_{xx}$$

as in Lecture 4, slide 6, we have to calculate $\Phi'(s)$:

$$\begin{aligned} \Phi'(s) &= \frac{d}{ds} \Psi'(s^2) s \\ &= \Psi''(s^2) 2s^2 + \Psi'(s^2)\ . \end{aligned} \tag{4}$$

Since Ψ is a penaliser function, we assume at least that it is increasing, i. e. $\Psi'(s^2) > 0$ for all $s \in \mathbb{R}$. If, as assumed, $\Psi(s^2)$ is convex in addition, we know that $\Psi''(s^2) > 0$, and the whole expression (4) is positive.

The diffusion process corresponding to regularisation with a convex penaliser thus can only perform forward diffusion. It has the possibility to be edge preserving, but not edge enhancing.