Differential Equations in Image Processing and Computer Vision 2008
Example Solutions for Theoretical Assignments 2 (T2)

Problem 1

(a) The Perona-Malik diffusivity $g(s^2) := \frac{1}{1+s^2/\lambda^2}$ leads to the flux:

$$\Phi(s) = s \left(1 + \frac{s^2}{\lambda^2}\right)^{-1} .$$

The derivative of the flux is

$$
\begin{aligned}
\Phi'(s) &= \left(1 + \frac{s^2}{\lambda^2}\right)^{-1} - s \left(1 + \frac{s^2}{\lambda^2}\right)^{-2} \frac{2s}{\lambda^2} \\
&= \left(1 + \frac{s^2}{\lambda^2}\right)^{-2} \left(1 + \frac{s^2}{\lambda^2} - \frac{2s^2}{\lambda^2}\right) \\
&= \left(1 + \frac{s^2}{\lambda^2}\right)^{-2} \frac{\lambda^2 - s^2}{\lambda^2}
\end{aligned}
$$

We note that

$$\Phi'(s) > 0 \quad \Longleftrightarrow \quad |s| < \lambda \qquad \text{(forward diffusion)}$$

and

$$\Phi'(s) < 0 \quad \Longleftrightarrow \quad |s| > \lambda \qquad \text{(backward diffusion) .}$$

That means, backward diffusion is possible for a local contrast $|u_x|$ larger than the contrast parameter λ. Edges with a larger contrast thus can be enhanced with isotropic linear diffusion with this diffusivity.

(b) For the Charbonnier diffusivity $g(s^2) := \frac{1}{\sqrt{1+s^2/\lambda^2}}$ we have

$$
\begin{aligned}
\Phi'(s) &= \frac{-2s^2}{2\lambda^2} \left(1 + \frac{s^2}{\lambda^2}\right)^{-\frac{3}{2}} + \left(1 + \frac{s^2}{\lambda^2}\right)^{-\frac{1}{2}} \\
&= \left(1 + \frac{s^2}{\lambda^2}\right)^{-\frac{3}{2}} \quad > 0 .
\end{aligned}
$$

In this case, the sign of Φ' is independent of s, and the Charbonnier diffusion process always performs forward diffusion.

(c) For the exponential Perona-Malik diffusivity $g(s^2) := \exp\left(\frac{-s^2}{2\lambda^2}\right)$ we obtain

$$\Phi'(s) = \left(1 - \frac{s^2}{\lambda^2}\right) \exp\left(\frac{-s^2}{2\lambda^2}\right) .$$

The resulting sign of $\Phi'(s)$ has the same behaviour than for the rational Perona-Malik seen above. We also have the possibility of backward diffusion at positions with high local contrast.

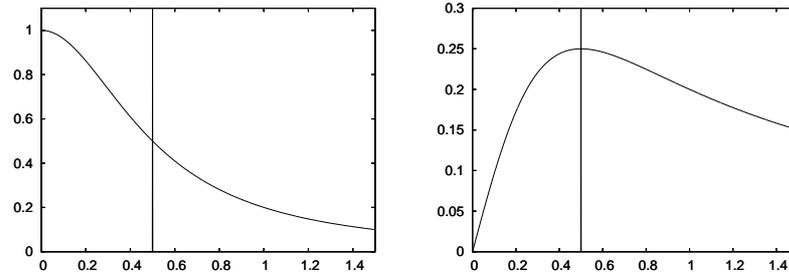The behaviour of the diffusivity and flux functions is displayed in the following figures:

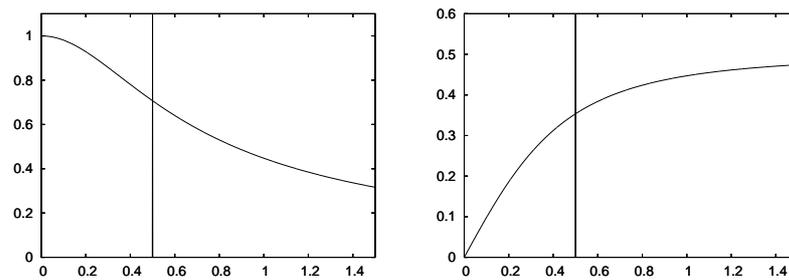Figure 1: Perona-Malik process. **Left:** diffusivity. **Right:** flux. The vertical line denotes the value of λ.

Figure 2: Charbonnier process. **Left:** diffusivity. **Right:** flux. The vertical line denotes the value of λ.
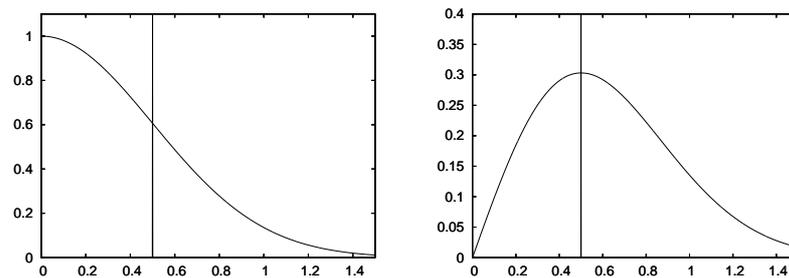

Figure 3: Exponential process. **Left:** diffusivity. **Right:** flux. The vertical line denotes the value of λ.

Problem 2

Given is the explicit scheme

$$\frac{u^{k+1} - u^k}{\tau} = A\left(u^k\right) u^k \qquad \left(\frac{\partial u}{\partial t} = A(u)u\right).$$

As we have seen in the beginning of Lecture 5, a spatial discretisation yields the following equation:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= A(u)u \\
\Leftrightarrow \frac{u_{ij}^{k+1} - u_{ij}^k}{\tau} &= \frac{1}{2h_1^2}\left((g_{i+1,j} + g_{ij})(u_{i+1,j}^k - u_{ij}^k) - (g_{ij} + g_{i-1,j})(u_{ij}^k - u_{i-1,j}^k)\right) \\
&+ \frac{1}{2h_2^2}\left((g_{i,j+1} + g_{ij})(u_{i,j+1}^k - u_{ij}^k) - (g_{ij} + g_{i,j-1})(u_{ij}^k - u_{i,j-1}^k)\right) \\
\Leftrightarrow u_{ij}^{k+1} &= \frac{\tau}{2h_1^2}\left(g_{i+1,j}u_{i+1,j}^k - g_{i+1,j}u_{ij}^k + g_{ij}u_{i+1,j}^k - g_{ij}u_{ij}^k\right) \\
&- \frac{\tau}{2h_1^2}\left(g_{ij}u_{ij}^k - g_{ij}u_{i-1,j}^k + g_{i,j-1}u_{ij}^k - g_{i,j-1}u_{i-1,j}^k\right) \\
&+ \frac{\tau}{2h_2^2}\left(g_{i,j+1}u_{i,j+1}^k - g_{i,j+1}u_{ij}^k + g_{ij}u_{i,j+1}^k - g_{ij}u_{ij}^k\right) \\
&- \frac{\tau}{2h_2^2}\left(g_{ij}u_{ij}^k - g_{ij}u_{i,j-1}^k + g_{i,j-1}u_{ij}^k - g_{i,j-1}u_{i,j-1}^k\right) + u_{ij}^k \\
\Leftrightarrow u_{ij}^{k+1} &= u_{ij}^k\left(1 - \tau\frac{g_{i+1,j} + 2g_{ij} + g_{i-1,j}}{2h_1^2} - \tau\frac{g_{i,j-1} + 2g_{ij} + g_{i,j+1}}{2h_2^2}\right) \\
&+ u_{i+1,j}^k\left(\tau\frac{g_{i+1,j} + g_{ij}}{2h_1^2}\right) + u_{i-1,j}^k\left(\tau\frac{g_{ij} + g_{i-1,j}}{2h_1^2}\right) \\
&+ u_{i,j+1}^k\left(\tau\frac{g_{i,j+1} + g_{ij}}{2h_2^2}\right) + u_{i,j-1}^k\left(\tau\frac{g_{ij} + g_{i,j-1}}{2h_2^2}\right).
\end{aligned}
$$

This leads to the following stencil notation, where the ordering is according to the one given in Lecture 3:

0	$\tau\dfrac{g_{i,j+1} + g_{ij}}{2h_2^2}$	0
$\tau\dfrac{g_{ij} + g_{i-1,j}}{2h_1^2}$	$1 - \tau\dfrac{g_{i+1,j} + 2g_{ij} + g_{i-1,j}}{2h_1^2} - \tau\dfrac{g_{i,j-1} + 2g_{ij} + g_{i,j+1}}{2h_2^2}$	$\tau\dfrac{g_{i+1,j} + g_{ij}}{2h_1^2}$
0	$\tau\dfrac{g_{ij} + g_{i,j-1}}{2h_2^2}$	0

Problem 3

At each time step of the semi-implicit scheme the following linear system of equations has to be solved:

$$\underbrace{\left(I - \tau A(u^k)\right)}_{B(u^k)} u^{k+1} = u^k$$

(a) In the case of a 1-D signal of size 4 the system matrix $B(u^k)$ has the form

$$B(u^k) = \begin{pmatrix} c_1 & x_1^+ & 0 & 0 \\ x_2^- & c_2 & x_2^+ & 0 \\ 0 & x_3^- & c_3 & x_3^+ \\ 0 & 0 & x_4^- & c_4 \end{pmatrix}$$

with

$$
\begin{aligned}
&& c_1 = 1 + \tau \tfrac{g_1+g_2}{2h_x^2} && x_1^+ = -\tau \tfrac{g_1+g_2}{2h_x^2} \\
x_2^- = -\tau \tfrac{g_1+g_2}{2h_x^2} && c_2 = 1 + \tau \tfrac{g_1+2g_2+g_3}{2h_x^2} && x_2^+ = -\tau \tfrac{g_2+g_3}{2h_x^2} \\
x_3^- = -\tau \tfrac{g_2+g_3}{2h_x^2} && c_3 = 1 + \tau \tfrac{g_2+2g_3+g_4}{2h_x^2} && x_3^+ = -\tau \tfrac{g_3+g_4}{2h_x^2} \\
x_4^- = -\tau \tfrac{g_3+g_4}{2h_x^2} && c_4 = 1 + \tau \tfrac{g_3+g_4}{2h_x^2}
\end{aligned}
$$

(b) In the case of a 2-D signal of size 3×4 the system matrix $B(u^k)$ has the form

$$B(u^k) = \left(\begin{array}{ccc|ccc|ccc|ccc}
c_1 & x_1^+ & 0 & y_1^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
x_2^- & c_2 & x_2^+ & 0 & y_2^+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_3^- & c_3 & 0 & 0 & y_3^+ & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline
y_4^- & 0 & 0 & c_4 & x_4^+ & 0 & y_4^+ & 0 & 0 & 0 & 0 & 0 \\
0 & y_5^- & 0 & x_5^- & c_5 & x_5^+ & 0 & y_5^+ & 0 & 0 & 0 & 0 \\
0 & 0 & y_6^- & 0 & x_6^- & c_6 & 0 & 0 & y_6^+ & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & y_7^- & 0 & 0 & c_7 & x_7^+ & 0 & y_7^+ & 0 & 0 \\
0 & 0 & 0 & 0 & y_8^- & 0 & x_8^- & c_8 & x_8^+ & 0 & y_8^+ & 0 \\
0 & 0 & 0 & 0 & 0 & y_9^- & 0 & x_9^- & c_9 & 0 & 0 & y_9^+ \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & y_{10}^- & 0 & 0 & c_{10} & x_{10}^+ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{11}^- & 0 & x_{11}^- & c_{11} & x_{11}^+ \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & y_{12}^- & 0 & x_{12}^- & c_{12}
\end{array} \right)$$

with

$$
\begin{aligned}
& c_1 = 1 + \tau \tfrac{g_1+g_2}{2h_x^2} + \tau \tfrac{g_1+g_4}{2h_y^2} && x_1^+ = -\tau \tfrac{g_1+g_2}{2h_x^2} && y_1^+ = -\tau \tfrac{g_1+g_4}{2h_y^2} \\
x_2^- = -\tau \tfrac{g_1+g_2}{2h_x^2} & \quad c_2 = 1 + \tau \tfrac{g_1+2g_2+g_3}{2h_x^2} + \tau \tfrac{g_2+g_5}{2h_y^2} && x_2^+ = -\tau \tfrac{g_2+g_3}{2h_x^2} && y_2^+ = -\tau \tfrac{g_2+g_5}{2h_y^2} \\
x_3^- = -\tau \tfrac{g_2+g_3}{2h_x^2} & \quad c_3 = 1 + \tau \tfrac{g_2+g_3}{2h_x^2} + \tau \tfrac{g_3+g_6}{2h_y^2} && && y_3^+ = -\tau \tfrac{g_3+g_6}{2h_y^2}
\end{aligned}
$$

$$
\begin{aligned}
y_4^- = -\tau \tfrac{g_1+g_4}{2h_y^2} & \quad c_4 = 1 + \tau \tfrac{g_4+g_5}{2h_x^2} + \tau \tfrac{g_1+2g_4+g_7}{2h_y^2} && x_4^+ = -\tau \tfrac{g_4+g_5}{2h_x^2} && y_4^+ = -\tau \tfrac{g_4+g_7}{2h_y^2} \\
y_5^- = -\tau \tfrac{g_2+g_5}{2h_y^2} \quad x_5^- = -\tau \tfrac{g_4+g_5}{2h_x^2} & \quad c_5 = 1 + \tau \tfrac{g_4+2g_5+g_6}{2h_x^2} + \tau \tfrac{g_2+2g_5+g_8}{2h_y^2} && x_5^+ = -\tau \tfrac{g_5+g_6}{2h_x^2} && y_5^+ = -\tau \tfrac{g_5+g_8}{2h_y^2} \\
y_6^- = -\tau \tfrac{g_3+g_6}{2h_y^2} \quad x_6^- = -\tau \tfrac{g_5+g_6}{2h_x^2} & \quad c_6 = 1 + \tau \tfrac{g_5+g_6}{2h_x^2} + \tau \tfrac{g_3+2g_6+g_9}{2h_y^2} && && y_6^+ = -\tau \tfrac{g_6+g_9}{2h_y^2} \\
y_7^- = -\tau \tfrac{g_4+g_7}{2h_y^2} & \quad c_7 = 1 + \tau \tfrac{g_7+g_8}{2h_x^2} + \tau \tfrac{g_4+2g_7+g_{10}}{2h_y^2} && x_7^+ = -\tau \tfrac{g_7+g_8}{2h_x^2} && y_7^+ = -\tau \tfrac{g_7+g_{10}}{2h_y^2} \\
y_8^- = -\tau \tfrac{g_5+g_8}{2h_y^2} \quad x_8^- = -\tau \tfrac{g_7+g_8}{2h_x^2} & \quad c_8 = 1 + \tau \tfrac{g_7+2g_8+g_9}{2h_x^2} + \tau \tfrac{g_5+2g_8+g_{11}}{2h_y^2} && x_8^+ = -\tau \tfrac{g_8+g_9}{2h_x^2} && y_8^+ = -\tau \tfrac{g_8+g_{11}}{2h_y^2} \\
y_9^- = -\tau \tfrac{g_6+g_9}{2h_y^2} \quad x_9^- = -\tau \tfrac{g_8+g_9}{2h_x^2} & \quad c_9 = 1 + \tau \tfrac{g_8+g_9}{2h_x^2} + \tau \tfrac{g_6+2g_9+g_{12}}{2h_y^2} && && y_9^+ = -\tau \tfrac{g_9+g_{12}}{2h_y^2} \\
y_{10}^- = -\tau \tfrac{g_7+g_{10}}{2h_y^2} & \quad c_{10} = 1 + \tau \tfrac{g_{10}+g_{11}}{2h_x^2} + \tau \tfrac{g_7+g_{10}}{2h_y^2} && x_{10}^+ = -\tau \tfrac{g_{10}+g_{11}}{2h_x^2} && \\
y_{11}^- = -\tau \tfrac{g_8+g_{11}}{2h_y^2} \quad x_{11}^- = -\tau \tfrac{g_{10}+g_{11}}{2h_x^2} & \quad c_{11} = 1 + \tau \tfrac{g_{10}+2g_{11}+g_{12}}{2h_x^2} + \tau \tfrac{g_8+g_{11}}{2h_y^2} && x_{11}^+ = -\tau \tfrac{g_{11}+g_{12}}{2h_x^2} && \\
y_{12}^- = -\tau \tfrac{g_9+g_{12}}{2h_y^2} \quad x_{12}^- = -\tau \tfrac{g_{11}+g_{12}}{2h_x^2} & \quad c_{12} = 1 + \tau \tfrac{g_{11}+g_{12}}{2h_x^2} + \tau \tfrac{g_9+g_{12}}{2h_y^2} && &&
\end{aligned}
$$

4

Problem 4

(a) In the first step, we derive a decomposition of the pentadiagonal 5×5-matrix

$$A := \begin{pmatrix} \gamma_1 & \delta_1 & \varepsilon_1 & 0 & 0 \\ \beta_1 & \gamma_2 & \delta_2 & \varepsilon_2 & 0 \\ \alpha_1 & \beta_2 & \gamma_3 & \delta_3 & \varepsilon_3 \\ 0 & \alpha_2 & \beta_3 & \gamma_4 & \delta_4 \\ 0 & 0 & \alpha_3 & \beta_4 & \gamma_5 \end{pmatrix}$$

as a product of an upper and a lower triangular matrix

$$L := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ b_1 & 1 & 0 & 0 & 0 \\ a_1 & b_2 & 1 & 0 & 0 \\ 0 & a_2 & b_3 & 1 & 0 \\ 0 & 0 & a_3 & b_4 & 1 \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} c_1 & d_1 & e_1 & 0 & 0 \\ 0 & c_2 & d_2 & e_2 & 0 \\ 0 & 0 & c_3 & d_3 & e_3 \\ 0 & 0 & 0 & c_4 & d_4 \\ 0 & 0 & 0 & 0 & c_5 \end{pmatrix} .$$

To this end, we calculate the product

$$LR = \begin{pmatrix} c_1 & d_1 & e_1 & 0 & 0 \\ b_1c_1 & b_1d_1 + c_2 & b_1e_1 + d_2 & e_2 & 0 \\ a_1c_1 & a_1d_1 + c_2b_2 & a_1e_1 + b_2d_2 + c_3 & b_2e_2 + d_3 & e_3 \\ 0 & a_2c_2 & a_2d_2 + c_3b_3 & a_2e_2 + b_3d_3 + c_4 & b_3e_3 + d_4 \\ 0 & 0 & a_3c_3 & a_3d_3 + c_4b_4 & a_3e_3 + b_4e_4 + c_5 \end{pmatrix}$$

This product has to be equal to the matrix A, and so we obtain the following rules for calculating the entries of L and R:

$$\begin{aligned} e_i &= \varepsilon_i \quad (i = 1, \ldots, 3) \\[6pt] d_1 &= \delta_1 \\ d_i &= \delta_i - b_{i-1}e_{i-1} \quad (i = 2, \ldots, 4) \\[6pt] c_1 &= \gamma_1 \\ c_2 &= \gamma_2 - b_1 d_1 \\ c_i &= \gamma_i - b_{i-1}d_{i-1} - a_{i-2}e_{i-2} \quad (i = 3, \ldots, 5) \\[6pt] b_1 &= \beta_1/c_1 \\ b_i &= \frac{\beta_i - a_{i-1}d_{i-1}}{c_i} \quad (i = 2, \ldots, 4) \\[6pt] a_i &= \frac{\alpha_i}{c_i} \quad (i = 1, \ldots, 3) \end{aligned}$$

It is necessary to assume that A is invertible, since otherwise divisions by zero can appear, and some of the coefficients will not be defined. To implement this scheme, one has to be careful with the order of the calculations such that only previously calculated values appear in the right-hand side.

(b) In the second step, we use the decomposition to solve a linear system of equations. We start by solving the system $Ly = f$:

$$\begin{aligned} y_1 &= f_1 \\ y_2 &= f_2 - b_1 y_1 \\ y_i &= f_i - b_{i-1}y_{i-1} - a_{i-2}y_{i-2} \quad (i = 3, 4, 5) \ . \end{aligned}$$

Then we solve the second system $Rx = y$:

$$\begin{aligned}
x_5 &= y_5/c_5 \\
x_4 &= \frac{y_4 - d_4 x_5}{c_4} \\
x_i &= \frac{y_i - d_i x_{i+1} - e_i x_{i+2}}{c_i} \quad (i = 3, 2, 1)
\end{aligned}$$

leading to the final result x.