

Problem 1.

a) To be proven:

$$(f * g)'(x) = (f' * g)(x) = (f * g')(x).$$

We show that $(f * g)'(x) = (f * g')(x)$:

$$\begin{aligned}(f * g)'(x) &= \frac{\partial}{\partial x} \left[\int_{-\infty}^{\infty} f(t)g(x-t) dt \right] \\ &= \int_{-\infty}^{\infty} f(t)g'(x-t) dt \\ &= (f * g')(x).\end{aligned}$$

Due to the *commutativity* of the convolution operation, we derive the remaining equality:

$$(f * g)'(x) = (g * f)'(x) = (g * f')(x) = (f' * g)(x).$$

b) We have

$$K_{\sigma}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}},$$

so with $\sigma = \sqrt{2t}$ it holds

$$K_{\sqrt{2t}}(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}.$$

Now we have

$$\begin{aligned}u_x &= \frac{\partial}{\partial x} K_{\sqrt{2t}}(x) \\ &= \frac{\partial}{\partial x} \left[\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \right] \\ &= \frac{1}{2\sqrt{\pi t}} \left(\frac{-2x}{4t} \right) e^{-\frac{x^2}{4t}} \\ &= -\frac{1}{4t\sqrt{\pi t}} x e^{-\frac{x^2}{4t}},\end{aligned}$$

and

$$\begin{aligned}
u_{xx} &= \frac{\partial^2}{\partial x^2} K_{\sqrt{2t}}(x) \\
&= -\frac{1}{4t\sqrt{\pi t}} \frac{\partial}{\partial x} \left[x e^{-\frac{x^2}{4t}} \right] \\
&= -\frac{1}{4t\sqrt{\pi t}} \left[e^{-\frac{x^2}{4t}} + x \left(\frac{-2x}{4t} \right) e^{-\frac{x^2}{4t}} \right] \\
&= -\frac{1}{4t\sqrt{\pi t}} \left(1 - \frac{x^2}{2t} \right) e^{-\frac{x^2}{4t}} \\
&= \frac{1}{8t^2\sqrt{\pi t}} (x^2 - 2t) e^{-\frac{x^2}{4t}},
\end{aligned}$$

and

$$\begin{aligned}
u_t &= \frac{\partial}{\partial t} K_{\sqrt{2t}}(x) \\
&= \frac{\partial}{\partial t} \left[\frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}} \right] \\
&= \frac{\partial}{\partial t} \left[\frac{1}{2\sqrt{\pi}} t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} t^{-1} \right] \\
&= \frac{1}{2\sqrt{\pi}} \left(-\frac{1}{2} t^{-\frac{3}{2}} e^{-\frac{x^2}{4t}} + t^{-\frac{1}{2}} e^{-\frac{x^2}{4t}} \left(-\frac{x^2}{4} (-t^{-2}) \right) \right) \\
&= \frac{1}{2\sqrt{\pi}} \left(-\frac{1}{2t\sqrt{t}} + \frac{1}{\sqrt{t}} \frac{x^2}{4t^2} \right) e^{-\frac{x^2}{4t}} \\
&= \frac{1}{8t^2\sqrt{\pi t}} (x^2 - 2t) e^{-\frac{x^2}{4t}},
\end{aligned}$$

so obviously the sought identity holds.

c) We have

$$\begin{aligned}
&\frac{\partial}{\partial t} (K_{\sqrt{2t}} * f)(x) \\
&\stackrel{\text{part a)}}{=} \left(\frac{\partial}{\partial t} (K_{\sqrt{2t}} * f) \right)(x) \\
&\stackrel{\text{part b)}}{=} \left(\frac{\partial^2}{\partial x^2} (K_{\sqrt{2t}} * f) \right)(x) \\
&\stackrel{\text{part a)}}{=} \frac{\partial^2}{\partial x^2} (K_{\sqrt{2t}} * f)(x).
\end{aligned}$$

By this, we have shown that $u_t = u_{xx}$. It remains to verify that

$$\lim_{t \rightarrow 0} u(x, t) = f(x).$$

This, however, is clear, since the more we shrink the convolution kernel, the more the original image is preserved:

$$\lim_{t \rightarrow 0} K_{\sqrt{2t}} = \delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases}$$

with $K_{\sigma} = (\delta * f)(x)$ with $u(x, 0) = (K_0 * f)(x)$.

Problem 2.

In order to determine the coefficients of the derivative mask, we have to perform a Taylor expansions for all points around the pixel f_i . This yields

$$\begin{aligned}
 f_{i-2} &= f_i - \frac{2}{1}hf'_i + \frac{4}{2}h^2f''_i - \frac{8}{6}h^3f'''_i + \frac{16}{24}h^4f''''_i - \frac{32}{120}h^5f_i^{(5)} + \frac{64}{720}h^6f_i^{(6)} + O(h^7) \\
 f_{i-1} &= f_i - \frac{1}{1}hf'_i + \frac{1}{2}h^2f''_i - \frac{1}{6}h^3f'''_i + \frac{1}{24}h^4f''''_i - \frac{1}{120}h^5f_i^{(5)} + \frac{1}{720}h^6f_i^{(6)} + O(h^7) \\
 f_i &= f_i \\
 f_{i+1} &= f_i + \frac{1}{1}hf'_i + \frac{1}{2}h^2f''_i + \frac{1}{6}h^3f'''_i + \frac{1}{24}h^4f''''_i + \frac{1}{120}h^5f_i^{(5)} + \frac{1}{720}h^6f_i^{(6)} + O(h^7) \\
 f_{i+2} &= f_i + \frac{2}{1}hf'_i + \frac{4}{2}h^2f''_i + \frac{8}{6}h^3f'''_i + \frac{16}{24}h^4f''''_i + \frac{32}{120}h^5f_i^{(5)} + \frac{64}{720}h^6f_i^{(6)} + O(h^7).
 \end{aligned}$$

Since we are interested in computing an approximation to the second derivative, we have to choose the parameters $\alpha_{-2}, \dots, \alpha_2$ in such a way that the following holds:

$$\begin{aligned}
 0f_i + 0f'_i + 1f''_i + 0f'''_i + 0f''''_i &= \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0f_i + \alpha_1f_{i+1} + \alpha_2f_{i+2} \\
 &\approx (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2)f_i \\
 &\quad + (-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2)hf'_i \\
 &\quad + (4\alpha_{-2} + \alpha_{-1} + \alpha_1 + 4\alpha_2)\frac{1}{2}h^2f''_i \\
 &\quad + (-8\alpha_{-2} - \alpha_{-1} + \alpha_1 + 8\alpha_2)\frac{1}{6}h^3f'''_i \\
 &\quad + (16\alpha_{-2} + \alpha_{-1} + \alpha_1 + 16\alpha_2)\frac{1}{24}h^4f''''_i
 \end{aligned}$$

This leads to the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 4 & 1 & 0 & 1 & 4 \\ -8 & -1 & 0 & 1 & 8 \\ 16 & 1 & 0 & 1 & 16 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{2}{h^2} \\ 0 \\ 0 \end{pmatrix}.$$

Its solution gives the weights

$$\begin{aligned}
 \alpha_{-2} &= \alpha_2 = -\frac{1}{12h^2} \\
 \alpha_{-1} &= \alpha_1 = \frac{16}{12h^2} \\
 \alpha_0 &= -\frac{30}{12h^2}
 \end{aligned}$$

which leads to the approximation

$$\frac{-f_{i-2} + 16f_{i-1} - 30f_i + 16f_{i+1} - f_{i+2}}{12h^2}.$$

Plugging this into the Taylor expansion gives

$$\begin{aligned}
& \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0f_i + \alpha_1f_{i+1} + \alpha_2f_{i+2} \\
= & \underbrace{(-1 + 16 - 30 + 16 - 1)}_{=0} \frac{1}{h^2} f_i \\
& + \underbrace{(2 - 16 + 16 - 2)}_{=0} \frac{1}{12} \frac{h}{h^2} f'_i \\
& + \underbrace{(-4 + 16 + 16 - 4)}_{=1} \frac{1}{12} \frac{1}{2} \frac{h^2}{h^2} f''_i \\
& + \underbrace{(8 - 16 + 16 - 8)}_{=0} \frac{1}{12} \frac{1}{6} \frac{h^3}{h^2} f'''_i \\
& + \underbrace{(-16 + 16 + 16 - 16)}_{=0} \frac{1}{12} \frac{1}{24} \frac{h^4}{h^2} f''''_i \\
& + \underbrace{(32 - 16 + 16 - 32)}_{=0} \frac{1}{12} \frac{1}{120} \frac{h^5}{h^2} f^{(5)}_i \\
& + \underbrace{(-64 + 16 + 16 - 64)}_{=96 \neq 0} \frac{1}{12} \frac{1}{720} \frac{h^6}{h^2} f^{(6)}_i \\
& + O(h^5) \\
= & f''_i - \frac{1}{90} h^4 f^{(6)}_i + O(h^5) \\
= & f''_i + O(h^4)
\end{aligned}$$

which shows that the order of consistency of the computed approximation is given by 4.