Prof. Dr. Joachim Weickert **Summer Term 2008**

Differential Equations in Image Processing and Computer Vision

Solutions to Self-Test Problems

Problem 1 (Nonlinear Isotropic Diffusion)

Suppose we are given the diffusivity $g(s^2) = \exp\Big(-\frac{cs^4}{\lambda^4}\Big)$ with a constant c in a nonlinear isotropic diffusion process for a evolving image u.

a) What is the associated flux function?

b) Sketch this function on the rectangular grid given below.

c) Determine the constant c such that the contrast parameter λ coincides with the boundary between forward and backward diffusion.

Given is the diffusivity $g\left(s^2\right) = \exp\left(-(cs^4)/\lambda^4\right)$.

To a)

The associated flux function is defined via

$$\Phi\left(u_x\right) = g\left(u_x^2\right) u_x\,.$$

With $s := u_x$ we obtain

$$\Phi\left(s\right) = s\exp\left(-\frac{cs^4}{\lambda^4}\right)$$

It is now recommended to proceed with part c) instead of part b)!

To c)

The following conditions determine the regions of forward and backward diffusion in the current case, respectively:

$$\begin{aligned} \Phi'\left(u_x\right) > 0 \quad &\text{for} \quad |u_x| < \lambda \quad \text{(forward diffusion)} \\ \Phi'\left(u_x\right) < 0 \quad &\text{for} \quad |u_x| > \lambda \quad \text{(backward diffusion)}\,. \end{aligned}$$

The boundary between these regions is determined by the condition $\Phi'\left(u_x\right) = 0$.

We first compute

$$\begin{aligned}\Phi'(s) &= \left[s \exp\left(-\frac{cs^4}{\lambda^4}\right)\right]' \\ &= \exp\left(-\frac{cs^4}{\lambda^4}\right) + s \exp\left(-\frac{cs^4}{\lambda^4}\right)\left(-\frac{4cs^3}{\lambda^4}\right) \\ &= \exp\left(-\frac{cs^4}{\lambda^4}\right)\left(1 - \frac{4cs^4}{\lambda^4}\right).\end{aligned}$$

This means, $\Phi'(s)$ is zero for

$$\begin{aligned} 0 &= 1 - \frac{4cs^4}{\lambda^4} \\ \Leftrightarrow \quad cs^4 &= \frac{1}{4}\lambda^4 . \end{aligned}$$

(Remark: The latter equation is fulfilled for $s = \lambda\left(\frac{1}{4c}\right)^{1/4}$ which we use in the sketch of part b).)

If λ shall coincide with the boundary between forward and backward diffusion, we have to set $s = \lambda$ in the above condition, yielding

$$c\lambda^4 = \frac{1}{4}\lambda^4 \quad \Leftrightarrow \quad c = \frac{1}{4}.$$

To b)

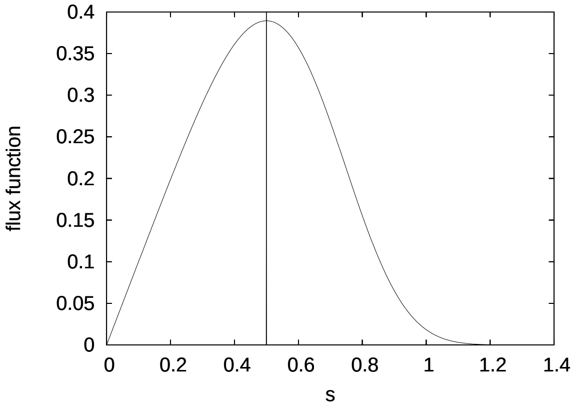


Abbildung 1: This Figure shows the flux function as well as a vertical line denoting the border point between forward and backward diffusion at $s = \lambda / (4c)^{1/4}$.

Problem 2 (Image Regularisation)

Consider the following energy functional for image regularisation:

$$E(u) = \int_{\Omega} \left(\beta\, \Psi\left((u-f)^2\right) + (1-\beta)\, u_x^2\right)\; dx \;\; ,$$

where $\beta \in [0, ..., 1]$.

a) Derive a discrete version of this energy functional that uses a vector $\mathbf{u} = (\mathbf{u}_1, \dots, \mathbf{u}_n)^T \in \mathbb{R}^n$ instead of an function u as an argument. Take the spatial grid size to be $h > 0$ and assume homogeneous Neumann boundary conditions.

b) For which task this discrete energy functional could also be used if β is not set fixed but allowed to vary dependent on the location x?

c) State the necessary conditions a vector $\mathbf{u} \in \mathbb{R}^n$ has to satisfy in order to minimize the discrete energy functional of part a).

Given is the functional

$$E(u) = \int_{\Omega} \left(\beta\Psi\left((u-f)^2\right) + (1-\beta)u_x^2\right)\, dx\,, \quad \beta \in [0,1]\,.$$

To a)

In a discrete version, the integral over Ω goes over to a corresponding sum, here from pixel 1 to pixel n, as given in the text. We need discrete versions of u and f, u_i and f_i, and we have to discretise u_x. The latter we do by a central difference.

As a quadrature of an integral

$$\int g\, dx$$

with a function g yields

$$\sum_i hg_i\,,$$

(midpoint-rule) where h is the pixel width, our formula will have the general form

$$E_h(u) = \sum_i hg_i\,.$$

Neglecting in a first step the boundary conditions, we obtain:

$$\tilde{E}_h(u) \;=\; \sum_{i=1}^{n} \left[h\beta\Psi\left((u_i - f_i)^2\right) + h\,(1-\beta)\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)^2 \right].$$

Thus, we seek a minimiser of

$$E_h(u) \;=\; \sum_{i=1}^{n} \left[\beta\Psi\left((u_i - f_i)^2\right) + (1-\beta)\left(\frac{u_{i+1}-u_{i-1}}{2h}\right)^2 \right].$$

Now we discuss the influence of homogeneous Neumann boundary conditions, i.e., $u_x = 0$. We add virtual points u_0 and u_{n+1} with

$$u_0 := u_1 \quad \text{and} \quad u_{n+1} := u_n\,.$$

The formula for $E_h(u)$ then works automatically.

To b)

Generalising the coefficient β in the Euler-Lagrange equations is useful for linear image interpolation.

To c)

Discrete Euler-Lagrange equations:

$$\begin{aligned}
\frac{\partial E_h}{\partial u_1} &= 2\beta\Psi'\left((u_1-f_1)^2\right)(u_1-f_1) - \frac{1-\beta}{2h^2}(u_3-u_1)\,,\\
\frac{\partial E_h}{\partial u_i} &= 2\beta\Psi'\left((u_i-f_i)^2\right)(u_i-f_i) + \frac{1-\beta}{2h^2}(u_i-u_{i-2}) - \frac{1-\beta}{2h^2}(u_{i+2}-u_i)\,,\; i=2,\,\ldots,\,n-1,\\
\frac{\partial E_h}{\partial u_n} &= 2\beta\Psi'\left((u_n-f_n)^2\right)(u_n-f_n) + \frac{1-\beta}{2h^2}(u_n-u_{n-2})\,.
\end{aligned}$$

Problem 3 (Motion Analysis)

Let a sufficiently often continuously differentiable 2-D image sequence $f(x, y, z)$ be given, where $z \geq 0$ denotes time. The goal is to formulate a model for estimating the 2-D optical flow $u(x, y, z)$ and $v(x, y, z)$ which allows an accurate estimation for *translational motion* under *varying illumination* and which is *robust against outliers* in the data term at the same time. Use a spatial smoothness term of your choice.

(a) Write down a suitable energy functional.

(b) Derive the differential equations which must necessarily be satisfied by a solution $u(x, y, z)$ and $v(x, y, z)$.
(Boundary conditions need *not* to be specified.)

To a)

Varying illumination implies, that the brightness constancy assumption is not recommended. Thus, we use the spatial brightness gradient constancy assumption. Using spatial homogeneous regularisation together with Gaussian window function K_ρ in the data term, robustifying the latter, yields:

$$E(u) = \int_\Omega \underline{u}^T \left(K_\rho * \left(\nabla_3 f_x \nabla_3 f_x^T\right) + K_\rho * \left(\nabla_3 f_y \nabla_3 f_y^T\right)\right) \underline{u} + \alpha \left(|\nabla_2 u|^2 + |\nabla_2 v|^2\right) \, dx \, ,$$

where $\alpha > 0$, and

$$\underline{u} = (u, v, 1)^T \; .$$

To b)

Before computing the Euler-Lagrange equations, we evaluate

$$\underline{u}^T \left(K_\rho * \left(\nabla_3 f_x \nabla_3 f_x^T\right) + K_\rho * \left(\nabla_3 f_y \nabla_3 f_y^T\right)\right) \underline{u} \, .$$

Let us emphasize, that the convolution by K_ρ affects only the matrices $\nabla_3 f_i \nabla_3 f_i^T$, $i = x, y$, componentwise, so we simply write $\hat{q}$ for convolved quantities q.

$$
\begin{aligned}
& (u,v,1)\, K_\rho * \begin{pmatrix} f_{xx}^2 & f_{xx}f_{xy} & f_{xx}f_{xz} \\ f_{xx}f_{xy} & f_{xy}^2 & f_{xy}f_{xz} \\ f_{xx}f_{xz} & f_{xy}f_{xz} & f_{xz}^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \\
& + (u,v,1)\, K_\rho * \begin{pmatrix} f_{yx}^2 & f_{yx}f_{yy} & f_{yx}f_{yz} \\ f_{yx}f_{yy} & f_{yy}^2 & f_{yy}f_{yz} \\ f_{yx}f_{yz} & f_{yy}f_{yz} & f_{yz}^2 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \\
= \; & (u,v,1) \begin{pmatrix} \widehat{f_{xx}^2} & \widehat{f_{xx}f_{xy}} & \widehat{f_{xx}f_{xz}} \\ \widehat{f_{xx}f_{xy}} & \widehat{f_{xy}^2} & \widehat{f_{xy}f_{xz}} \\ \widehat{f_{xx}f_{xz}} & \widehat{f_{xy}f_{xz}} & \widehat{f_{xz}^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \\
& + (u,v,1) \begin{pmatrix} \widehat{f_{yx}^2} & \widehat{f_{yx}f_{yy}} & \widehat{f_{yx}f_{yz}} \\ \widehat{f_{yx}f_{yy}} & \widehat{f_{yy}^2} & \widehat{f_{yy}f_{yz}} \\ \widehat{f_{yx}f_{yz}} & \widehat{f_{yy}f_{yz}} & \widehat{f_{yz}^2} \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \\
= \; & (u,v,1) \begin{pmatrix} \widehat{f_{xx}^2}u + \widehat{f_{xx}f_{xy}}v + \widehat{f_{xx}f_{xz}} \\ \widehat{f_{xx}f_{xy}}u + \widehat{f_{xy}^2}v + \widehat{f_{xy}f_{xz}} \\ \widehat{f_{xx}f_{xz}}u + \widehat{f_{xy}f_{xz}}v + \widehat{f_{xz}^2} \end{pmatrix} \\
& + (u,v,1) \begin{pmatrix} \widehat{f_{yx}^2}u + \widehat{f_{yx}f_{yy}}v + \widehat{f_{yx}f_{yz}} \\ \widehat{f_{yx}f_{yy}}u + \widehat{f_{yy}^2}v + \widehat{f_{yy}f_{yz}} \\ \widehat{f_{yx}f_{yz}}u + \widehat{f_{yy}f_{yz}}v + \widehat{f_{yz}^2} \end{pmatrix} \\
= \; & \widehat{f_{xx}^2}u^2 + 2\widehat{f_{xx}f_{xy}}uv + 2\widehat{f_{xx}f_{xz}}u + \widehat{f_{xy}^2}v^2 + 2\widehat{f_{xy}f_{xz}}v + \widehat{f_{xz}^2} \\
& + \widehat{f_{yx}^2}u^2 + 2\widehat{f_{yx}f_{yy}}uv + 2\widehat{f_{yx}f_{yz}}u + \widehat{f_{yy}^2}v^2 + 2\widehat{f_{yy}f_{yz}}v + \widehat{f_{yz}^2}\,.
\end{aligned}
$$

The Euler-Lagrange equations we want to compute read (without boundary conditions):

$$0 = F_u - \frac{\partial}{\partial x} F_{u_x} - \frac{\partial}{\partial y} F_{u_y} \tag{1}$$

$$0 = F_v - \frac{\partial}{\partial x} F_{v_x} - \frac{\partial}{\partial y} F_{v_y}\,. \tag{2}$$

We now compute the ingredients of the above equations:

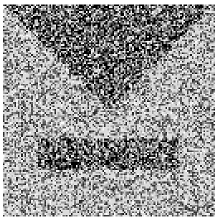
$$
\begin{aligned}
F_u &= 2\widehat{f_{xx}^2}u + 2\widehat{f_{xx}f_{xy}}v + 2\widehat{f_{xx}f_{xz}} + 2\widehat{f_{yx}^2}u + 2\widehat{f_{yx}f_{yy}}v + 2\widehat{f_{yx}f_{yz}}\,, \\
F_{u_x} &= 2\alpha u_x\,, \\
F_{u_y} &= 2\alpha u_y\,, \\
F_v &= 2\widehat{f_{xy}^2}v + 2\widehat{f_{xx}f_{xy}}u + 2\widehat{f_{xy}f_{xz}} + 2\widehat{f_{yy}^2}v + 2\widehat{f_{yx}f_{yy}}u + 2\widehat{f_{yy}f_{yz}}\,, \\
F_{v_x} &= 2\alpha v_x\,, \\
F_{v_y} &= 2\alpha v_y\,.
\end{aligned}
$$

Plugging these formulae together yields the Euler-Lagrange equations

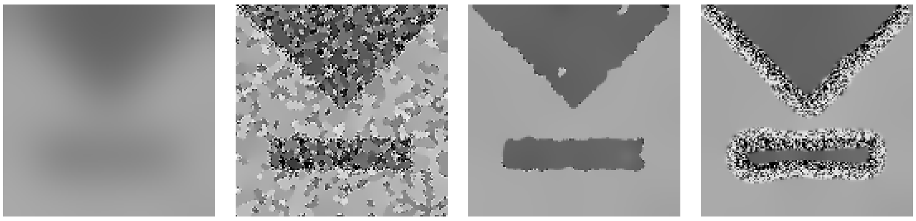
$$
\begin{aligned}
0 &= \hat{f}^2_{xx} u + \widehat{f_{xx} f_{xy}} v + \widehat{f_{xx} f_{xz}} + \hat{f}^2_{yx} u + \widehat{f_{yx} f_{yy}} v + \widehat{f_{yx} f_{yz}} - \alpha \Delta u \,, \\
0 &= \hat{f}^2_{xy} v + \widehat{f_{xx} f_{xy}} u + \widehat{f_{xy} f_{xz}} + \hat{f}^2_{yy} v + \widehat{f_{yx} f_{yy}} u + \widehat{f_{yy} f_{yz}} - \alpha \Delta v \,.
\end{aligned}
$$

Problem 4 (Visual Analysis)

The images **A**–**D** show results of regularised nonlinear isotropic diffusion filtering applied to the original image **O**.



O



A **B** **C** **D**

The parameter values are – not necessarily in the same order –

(i) $\lambda = 3.0$, $\sigma = 1.3$, **(ii)** $\lambda = 1.0$, $\sigma = 5.0$,

(iii) $\lambda = 7.0$, $\sigma = 1.3$, **(iv)** $\lambda = 7.0$, $\sigma = 5.0$

where σ is the smoothing parameter, and λ the threshold parameter from the diffusivity function $g(|\nabla u_\sigma|^2) = 1 - \exp\left(-3.31488/(|\nabla u_\sigma|/\lambda)^8\right)$. In all images, 500 iterations of time step size 0.2 were performed.

Indicate for each of the parameter settings (i)–(iv) which of the filtered images A–D corresponds.

The correct correspondences are

(i) $\lambda = 3.0$, $\sigma = 1.3$ **B**, **(ii)** $\lambda = 1.0$, $\sigma = 5.0$ **D**,

(iii) $\lambda = 7.0$, $\sigma = 1.3$ **C**, **(iv)** $\lambda = 7.0$, $\sigma = 5.0$ **A**.

Problem 5 (Connection between Perona-Malik and Self-Snakes)

Let the isotropic nonlinear diffusion process of Perona-Malik

$$u_t = \operatorname{div}\left(g(|\nabla u|^2)\, \nabla u\right)$$

and the self-snake process

$$v_t = |\nabla v| \operatorname{div}\left(g(|\nabla v|^2)\, \frac{\nabla v}{|\nabla v|}\right)$$

be given. Show that the following relation holds:

$$u_t - v_t = g(|\nabla u|^2)\, u_{\eta\eta}\ .$$

Solution:

$$\begin{aligned}
u_t - v_t &= \operatorname{div}\left(g(|\nabla u|^2)\, \nabla u\right) - |\nabla u| \operatorname{div}\left(g(|\nabla u|^2)\, \frac{\nabla u}{|\nabla u|}\right) \\
&= g(|\nabla u|^2)\, \underbrace{\Delta u}_{u_{\xi\xi} + u_{\eta\eta}} + \nabla g(|\nabla u|^2)^\top \nabla u - g(|\nabla u|^2)\, \underbrace{|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)}_{u_{\xi\xi}} - \nabla g(|\nabla u|^2)^\top \nabla u \\
&= g(|\nabla u|^2)\, (u_{\xi\xi} + u_{\eta\eta}) - g(|\nabla u|^2)\, u_{\xi\xi} \\
&= g(|\nabla u|^2)\, u_{\eta\eta}
\end{aligned}$$

Problem 6 (Multiple Choice)

Answer the following questions by writing Y (yes) or N (no) in the corresponding box to the right.

a) The AOS scheme is invariant under rotations. Y

b) Pure backward diffusion processes are ill-posed problems. Y

c) An optic flow data term based on constany assumptions on two different constancy assumptions is sufficient to overcome the aperture problem locally (without the help of the smoothness term). N

d) A dilation step followed by an erosion step erases local maxima. N

e) The number of extrema does not decrease during a 1-D linear diffusion process. N

f) Anisotropic diffusion cannot be extended to matrix-valued images, since the diffusion tensor is a matrix itself. N