Lecture 23:

Segmentation II:

Optimisation Methods

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Disadvantages of Previous Segmentation Methods

◆ selection of multiple parameters (e.g. for double thresholding)

◆ sometimes less favourable results:

  • unsmooth region boundaries (e.g. for watersheds)
  • noise and too small structures create their own regions (e.g. for region merging)

◆ removal of less favourable results requires postprocessing steps (and hence new parameters)

◆ no tool for comparing the quality of different segmentations

Is there a systematic approach that avoids these shortcomings and requires only one single parameter (scale parameter)?

The Mumford–Shah Cartoon Model

◆ Let $f : \Omega \to \mathbb{R}$ be a continuous initial image.

◆ goal: segmentation (u, K), where u is constant in each segment and K represents the set of segment boundaries ("cartoon").

◆ simple variational ansatz: segmentation as minimiser of the energy functional

$$E(u, K) = \int\limits_{\Omega \setminus K} (u - f)^2 \, dx \, dy \; + \; \lambda \, l(K),$$

where $l(K)$ measures the length of the segment boundaries and $\lambda > 0$ is a scale parameter.

◆ Term 1 contols the homogeneity within a segment, while term 2 optimises the length of the boundaries. Increasing λ makes it more attractive to have larger segments (at the expense of homogeneity).

◆ sufficiently complicated to represent basic features of a good segmentation; sufficiently simple to allow a thorough mathematical analysis

Theoretical Properties

◆ **Preservation of Mean**
Term 1 implies that, within one segment, u has the same mean as f.
Thus, given f, the segmentation is uniquely determined by the boundary set K, and we may write $E(K)$ instead of $E(u, K)$.

◆ **Existence of a Global Minimiser, Regularity of the Boundaries**
There exists (at least) one segmentation that minimises $E(K)$.
The segment boundaries are either regular (at least C^1),
or they are singular and of one of the following types:

- Three edges meet at an angle of 120 degrees.
- An edge meets the boundary of the image domain (at 90 degrees).

◆ **Many Local Minimisers**
The energy landscape may have numerous local minimisers. There are two options:

- very slow minimisation methods that guarantee to find the global minimum asymptotically *(simulated annealing)*.
- more efficient approximations that usually find reasonable local minima but cannot guarantee to end up in a global minimum

An Approximation Method for Minimisation

In order to find good local minima of $E(u, K)$ in an efficient way, consider so-called 2-normal segmentations.

Definition (2-normal segmentation):

A segmentation K is called *2-normal*, if every segmentation K', that results from merging two adjacent segments of K, has a larger energy:

$$E(K') \; > \; E(K).$$

Restricting ourselves to 2-normal segmentations allows to prove a number of useful properties.

What are the Properties of 2-Normal Segmentations ?

◆ **Limitation of the Number of Regions**

$$n \; \leq \; c_0(\Omega) \, \frac{|\Omega| \, (\sup f - \inf f)^4}{\lambda^2}$$

where $|\Omega|$ denotes the area of the image domain Ω, and c_0 is a known constant that depends only on the geometry of Ω.
This shows:

- The larger the domain Ω, the more segments can be expected.
- High image contrast encourages many segments.
- Increasing λ encourages less segments.

◆ **Elimination of Small Regions**

The area of each segment Ω_i, $i = 1,...,n$ is bounded from below:

$$|\Omega_i| \geq c_1(f, \lambda, \Omega).$$

◆ **Elimination of Thin Regions**

The length $l(\partial\Omega_i)$ of a segment boundary $\partial\Omega_i$ cannot become arbitrary large compared to the square root of its area:

$$l(\partial\Omega_i) \leq c_2(f, \Omega)\,|\Omega_i|^{1/2}$$

Since the segment area $|\Omega_i|$ is bounded by $|\Omega|$, there exists an upper limit for the length of a segment boundary.

◆ **Smoothness of the Segment Boundaries**

The segment boundaries are C^1 almost everywhere.

Algorithmic Realisation

Basic Idea:

◆ approximate global minimiser of cartoon model by local minimiser resulting from the restriction to 2-normal segmentations

◆ create 2-normal segmentations by region merging

◆ perform region merging by successively increasing the scale parameter λ

Useful Observation

◆ Merging two segments Ω_i, Ω_j with means u_i, u_j does not require to recompute the entire energy functional.

◆ One can show that the new energy is given by

$$E(K - \partial(\Omega_i, \Omega_j)) = E(K) + \frac{|\Omega_i| \cdot |\Omega_j|}{|\Omega_i| + |\Omega_j|}(u_i - u_j)^2 - \lambda\, l(\partial(\Omega_i, \Omega_j))$$

where $\partial(\Omega_i, \Omega_j)$ is the joint border between Ω_i and Ω_j.

The Region Merging Algorithm

1. Initialisation: Each pixel is one segment.

2. For each pair of neighbouring pixels, one computes the λ value, for which a merging decreases the energy.

3. Merge the pair with the smallest λ value.

4. Compute the λ values for merging the new segment with its neighbouring segments.

5. Repeat the steps (2)–(4), until a desired number of regions or a specified value for λ is reached.

Important Properties

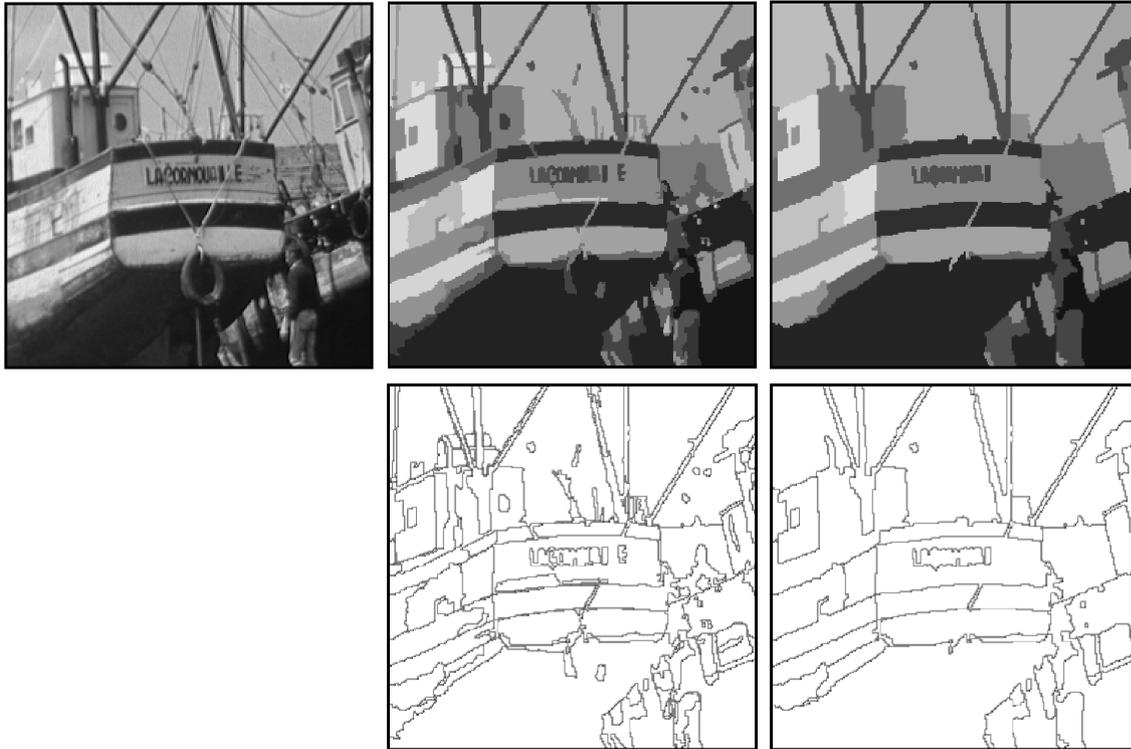◆ *Fixed Point Property:*
For a piecewise constant image there exists some λ_0 such that it is a valid segmentation for all $\lambda < \lambda_0$.

◆ *Pyramid Property, Causality:*
For $\lambda_2 > \lambda_1$, the boundaries of a λ_2-segmentation are a subset of the boundaries of a λ_1-segmentation.

Experiments
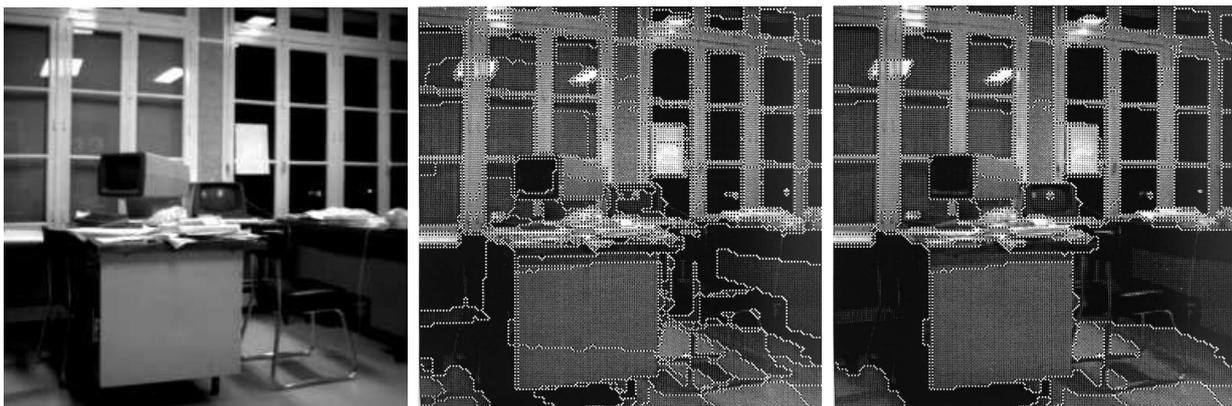
(a) Left: Original image. **(b) Middle:** Segmented with $\lambda = 1024$. **(c) Right:** $\lambda = 4096$. Authors: G. Koepfler, J.-M. Morel, S. Solimini (1990).

(a) Top left: Original image. **(b) Top middle:** Segmented with $\lambda = 2022$. **(c) Top right:** Segmented with $\lambda = 6173$. **(d) Bottom middle:** Segmentation boundaries of (b). **(e) Bottom right:** Segmentation boundaries of (c). Author: G. Koepfler (1995).

The Active Contour Model of Chan and Vese

◆ Chan and Vese considered the cartoon model under the restriction that the image is segmented into two phases only.

◆ The energy functional then becomes a function of a contour C:

$$E(C) = \int_{\text{inside } C} (f(x,y) - u_1)^2 \, dx \, dy \; + \; \int_{\text{outside } C} (f(x,y) - u_2)^2 \, dx \, dy \; + \; \lambda \, l(C)$$

where u_1 and u_2 are the arithmetic means of $f(x,y)$ inside/outside C.

◆ The user specifies some initial contour C_0 close to the segmentation one wants to find.

◆ Minimising $E(C)$ drives the curve towards the segment boundaries (using so-called level set methods that even allow a contour to split; we omit the technical details).

Top left to bottom right: Evolution of an active contour under the Chan-Vese model. The contour splits and creates a fairly good segmentation of Europe at night. Authors: T. Chan, L. Vese (2001).

Extensions of the Chan–Vese Model

◆ recent modifications of the Chan–Vese model include

  • more sophisticated features (colour, texture, motion, ...)

  • additional statistical characterisations of a region
    (not only mean, but also standard deviation, ...)

  • multiphase models that create more than two types of segments

  • sometimes even a-priori knowledge using a statistical characterisation of the
    shapes to be expected (Lecture 30)

◆ yield state-of-the-art results in segmentation
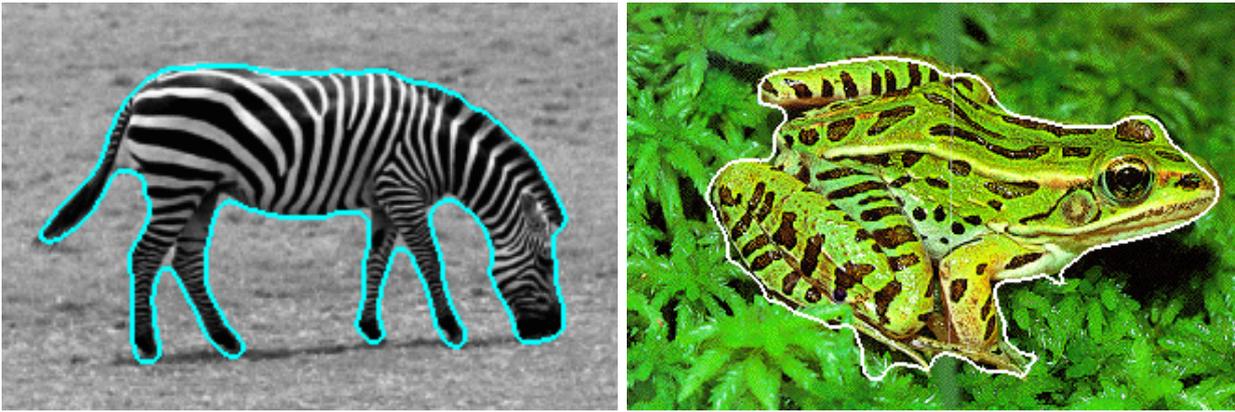
(a) Left: Segmented image using grey values and texture as features. **(b) Right:** Segmentation with colour and texture as feature channels. Author: T. Brox (2003).

Summary

◆ The cartoon model of Mumford and Shah is an energy based segmentation method that rewards homogeneity within each region and a small length of segmentation boundaries.

◆ requires only a single parameter that is related to the scale

◆ relatively simple and mathematically well understood

◆ restriction to 2-normal segmentations automatically generates many desirable features that have to be engineered in an ad hoc way for most other methods: The segments are neither too many, too small, too thin, nor do they have too irregular boundaries.

◆ algorithmic realisation via region merging

◆ Active contour variants exist that allow interactive segmentation of high quality.

Summary (2)

Literature

◆ G. Koepfler: *Regionenwachstumsverfahren und Variationsmodelle*. Vortragsausarbeitung (Heidelberg, Nov. 1995). www.math-info.univ-paris5.fr/~gk/papers/heidelberg95.pdf
(well-readable overview)

◆ J.-M. Morel, S. Solimini: Segmentation of images by variational methods: a constructive approach. *Revista Matematica de la Universidad Complutense de Madrid*, Vol. 1, No. 1–3, pp. 169–182, 1988.
(theoretical analysis of the cartoon model)

◆ G. Koepfler, C. Lopez, J.-M. Morel: A multiscale algorithm for image segmentation by variational method. *SIAM Journal on Numerical Analysis*, Vol. 31, pp. 282–299, 1994.
(2-normal segmentation by region merging)

◆ J.-M. Morel, S. Solimini: *Variational Methods in Image Segmentation*. Birkhäuser, Basel, 1994.
(an entire book on Mumford-Shah segmentation)

◆ D. Mumford, J. Shah: Boundary detection by minimizing functionals, I. In *Proc. IEEE Computer Society Conference on Computer Vision and Pattern Recognition* (San Francisco, CA, June 1985), pp. 22–26, 1985.
(introduced the Mumford-Shah functional)

◆ D. Mumford, J. Shah: Optimal approximation by piecewise smooth functions and associated variational problems. *Communications in Pure and Applied Mathematics*, Vol. 42, pp. 577–685, 1989.
(very detailed mathematical analysis)

Summary (3)

◆ T. F. Chan, L. A. Vese: Active contours without edges. *IEEE Transactions on Image Processing*, Vol. 10, No. 2, pp. 266–277, Febr. 2001.
(Chan–Vese model)

◆ T. Brox, M. Rousson, R. Deriche, J. Weickert: Unsupervised segmentation incorporating colour, texture, and motion. In N. Petkov, M. A. Westenberg: *Computer Analysis of Images and Patterns*. Springer LNCS Vol. 2756, pp. 353–360, Berlin, 2003.
(extensions of the Chan–Vese model)