

Lecture 16: Image Enhancement VII: Continuous Variational Methods

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Motivation

Motivation

- ◆ In the last lecture we considered discrete variational models of type

$$E_f(\mathbf{u}) := \frac{1}{2} \sum_{k=1}^N \left((u_k - f_k)^2 + \frac{\alpha}{2} \sum_{l \in \mathcal{N}(k)} (u_l - u_k)^2 \right).$$

- ◆ Today we consider continuous counterparts.
- ◆ The continuous formulation can offer some advantages:
 - Some continuous models are more transparent and elegant.
 - One continuous model may give rise to a number of discrete models (depending on the specific discretisation).
 - Invariance under rotation is easily satisfied in the continuous setting.
- ◆ In order to minimise a continuous energy, however, we have to learn some mathematics from the calculus of variations.

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A Continuous Variational Method

- ◆ consider 2-D setting with a continuous image $f : \Omega \rightarrow \mathbb{R}$ with $\Omega \subset \mathbb{R}^2$
- ◆ smoothed image $u : \Omega \rightarrow \mathbb{R}$ as minimiser of

$$E_f(u) := \frac{1}{2} \int_{\Omega} \left((u - f)^2 + \alpha |\nabla u|^2 \right) dx dy$$

with $\nabla u := (u_x, u_y)^\top$.

- ◆ The minimising image u is no longer a vector like in the discrete case, but a function.

How is this function obtained?

To this end, we make an excursion to the calculus of variations.

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Calculus of Variations

Calculus of Real Numbers (reellwertige Analysis):

- ◆ considers real-valued *functions* $f(x)$: mappings from a *real number* x to a real number
- ◆ If f has a minimum in ξ , then ξ necessarily satisfies $f'(\xi) = 0$.
- ◆ If f is strictly convex, then ξ is the unique minimum.

Calculus of Variations (Variationsrechnung):

- ◆ considers real-valued *functionals* $E(u)$: mappings from a *function* $u(x)$ to a real number
- ◆ If E is minimised by a function v , then v necessarily satisfies the corresponding *Euler-Lagrange equation*, a differential equation in v .
- ◆ If E is strictly convex, then v is the unique minimiser.

Calculus of Variations (2)

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The mathematicians Euler and Lagrange belong to the founders of the calculus of variations. **Left:** Leonhard Euler (1707–1783). **Right:** Joseph-Louis Lagrange (1736–1813). Source: www-gap.dcs.st-and.ac.uk/~history/PictDisplay/.

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Calculus of Variations (3)

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Theorem (Euler–Lagrange Equation in 1-D):

A smooth function $u(x)$, $x \in [a, b]$ that minimises the functional

$$E(u) = \int_a^b F(x, u, u') dx$$

necessarily satisfies the *Euler–Lagrange equation*

$$F_u - \frac{d}{dx} F_{u'} = 0$$

with so-called *natural boundary conditions*

$$F_{u'} = 0$$

in $x = a$ and $x = b$.

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Calculus of Variations (4)



Proof

We assume that $v(x)$ is a sufficiently often differentiable minimiser of E . We embed $v(x)$ into the family

$$u(x, \varepsilon) := v(x) + \varepsilon h(x)$$

with some perturbation function $h(x)$.

Since $v(x)$ minimises $E(u)$, we know that the scalar-valued function

$$g(\varepsilon) := E(u(x, \varepsilon)) = E(v + \varepsilon h)$$

has a minimum in $\varepsilon = 0$. Therefore, we have

$$0 = g'(0) = \frac{d}{d\varepsilon} E(v + \varepsilon h) \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_a^b F(x, \underbrace{v + \varepsilon h}_{u(\cdot, \varepsilon)}, \underbrace{v' + \varepsilon h'}_{u'(\cdot, \varepsilon)}) dx \Big|_{\varepsilon=0}.$$

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Calculus of Variations (5)



By means of the chain rule it follows that

$$\begin{aligned} 0 &= \int_a^b \left(F_u(x, u, u') h(x) + F_{u'}(x, u, u') h'(x) \right) dx \Big|_{\varepsilon=0} \\ &= \int_a^b \left(F_u(x, v, v') h(x) + F_{u'}(x, v, v') h'(x) \right) dx. \end{aligned}$$

Partial integration of the second term yields

$$0 = \int_a^b \left(F_u(x, v, v') h(x) - \frac{d}{dx} F_{u'}(x, v, v') h(x) \right) dx + F_{u'}(x, v, v') h(x) \Big|_{x=a}^{x=b} \quad (1)$$

If v is a minimiser within some family of competing functions $u(\varepsilon, x) = v(x) + \varepsilon h(x)$, then v is also minimiser within the smaller class of functions where the perturbation $h(x)$ satisfies $h(a) = 0 = h(b)$. Thus,

$$0 = \int_a^b \left(F_u(x, v, v') - \frac{d}{dx} F_{u'}(x, v, v') \right) h(x) dx.$$

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Calculus of Variations (6)



This gives the Euler–Lagrange equation

$$F_u - \frac{d}{dx} F_{u'} = 0.$$

Note that this equation does not depend on the perturbation h .
Plugging it into Equation (1) gives

$$F_{u'}(x, v, v') h(x) \Big|_{x=a}^{x=b} = 0.$$

which holds for arbitrary perturbations h (also with $h(a) \neq 0$ and $h(b) \neq 0$).
Thus, one obtains the natural boundary conditions

$$F_{u'} = 0$$

for $x = a$ und $x = b$. This concludes the proof.

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Calculus of Variations (7)



Extensions to the 2-D Case

$$E(u) = \int_{\Omega} F(x, y, u, u_x, u_y) dx dy$$

yields the Euler–Lagrange equation

$$F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y} = 0$$

with the natural boundary condition

$$\mathbf{n}^{\top} \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} = 0$$

on the image boundary $\partial\Omega$ with normal vector \mathbf{n} .

Extensions to higher dimensions are analog.

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Extensions to Vector-Valued Functions

$$E(u, v) = \int_a^b F(x, u, v, u', v') dx$$

creates a set of Euler–Lagrange equations:

$$F_u - \frac{d}{dx} F_{u'} = 0,$$

$$F_v - \frac{d}{dx} F_{v'} = 0$$

with natural boundary conditions for u and v .

This will be needed in a later lecture on optic flow computation.

Extensions to vector-valued functions with more components are straightforward.

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Application to the Continuous Model (1)

Application to the Continuous Model

◆ For the integrand

$$F(x, y, u, u_x, u_y) := \frac{1}{2} (u - f)^2 + \frac{\alpha}{2} (u_x^2 + u_y^2)$$

we obtain the partial derivatives

$$F_u = u - f,$$

$$F_{u_x} = \alpha u_x,$$

$$F_{u_y} = \alpha u_y$$

◆ This leads to the Euler–Lagrange equation

$$\begin{aligned} 0 &= F_u - \frac{d}{dx} F_{u_x} - \frac{d}{dy} F_{u_y} \\ &= u - f - \frac{d}{dx} (\alpha u_x) - \frac{d}{dy} (\alpha u_y) \\ &= u - f - \alpha \Delta u \end{aligned}$$

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Application to the Continuous Model (2)



- ◆ The natural boundary condition

$$0 = \mathbf{n}^\top \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix}$$

on the image boundary $\partial\Omega$ gives

$$0 = \mathbf{n}^\top \nabla u = \partial_{\mathbf{n}} u$$

where we divided by α .

- ◆ Such a vanishing normal derivative at the boundaries of the image domain comes down to reflecting boundary conditions (or an extension by mirroring).
- ◆ The Euler–Lagrange equation contains partial derivative of the unknown function $u(x, y)$. Therefore it is a partial differential equation. Usually such equations have to be solved numerically.
- ◆ The discrete model from the previous lecture can be regarded as such a numerical approximation. Thus, we only need solvers for linear systems of equations.

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Relations to Discrete Variational Methods (1)



Relations to Discrete Variational Methods

- ◆ For simplicity, consider the 1-D case.
- ◆ In the previous lecture we obtained for the 1-D discrete case:

$$0 = u_1 - f_1 - \alpha(u_2 - u_1),$$

$$0 = u_i - f_i - \alpha(u_{i+1} - 2u_i + u_{i-1}) \quad (i = 2, \dots, N-1),$$

$$0 = u_N - f_N - \alpha(-u_N + u_{N-1}).$$

- ◆ This may be regarded as a discretisation of the 1-D Euler–Lagrange equation

$$0 = u - f - \alpha u''$$

for the energy functional

$$E(u) = \frac{1}{2} \int_a^b ((u - f)^2 + \alpha(u')^2) dx$$

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Relations to Discrete Variational Methods (2)

- ◆ In this case, u'' was approximated in pixel i in the following way:

$$\begin{aligned} u''|_i &\approx \frac{u'_{i+1/2} - u'_{i-1/2}}{k} \\ &\approx \frac{\frac{u_{i+1} - u_i}{k} - \frac{u_i - u_{i-1}}{k}}{k} \\ &= \frac{u_{i+1} - 2u_i + u_{i-1}}{k^2} \end{aligned}$$

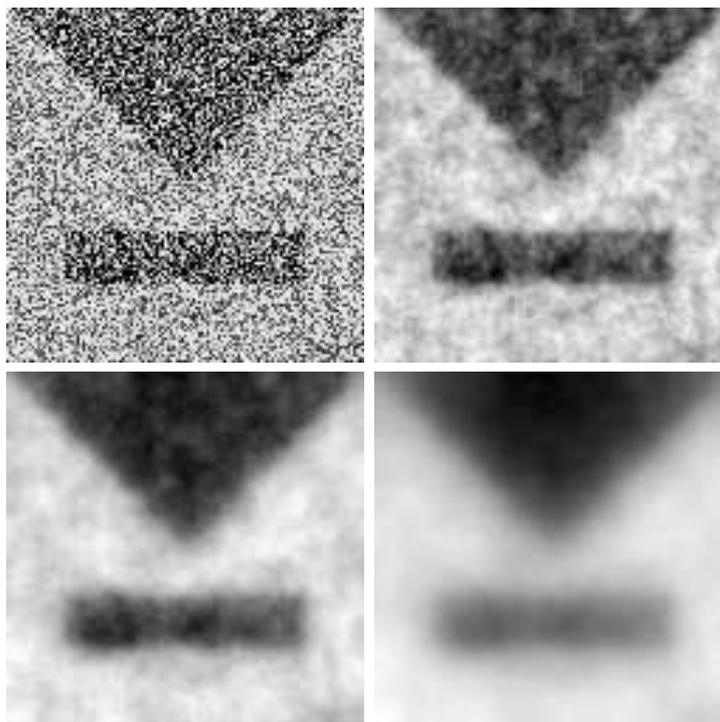
with spatial grid size (pixel distance) $k = 1$.

- ◆ The natural boundary conditions $u'(a) = 0$ and $u'(b) = 0$ were discretised by mirrored dummy boundary pixels $u_0 := u_1$ and $u_{N+1} := u_N$:

$$\begin{aligned} 0 = u'_{1/2} &\approx \frac{u_1 - u_0}{k} \\ 0 = u'_{N+1/2} &\approx \frac{u_{N+1} - u_N}{k} \end{aligned}$$

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Relations to Discrete Variational Methods (3)



(a) **Top left:** Test image, 128×128 pixels. (b) **Top right:** Variational method with $\alpha = 5$. (c) **Bottom left:** $\alpha = 20$. (d) **Bottom right:** $\alpha = 100$. For improving visibility, an affine greyscale transformation to $[0, 255]$ has been performed. Author: J. Weickert (2000).

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Nonquadratic Variational Methods

Problem:

- ◆ Methods discussed so far blur edges.

Remedy:

- ◆ Modify the smoothness term $|\nabla u|^2$ such that large gradients are less severely penalised than before.
- ◆ Replace $|\nabla u|^2$ by a function $\Psi(|\nabla u|^2)$ that is still convex in $|\nabla u|$, but for large gradients it increases slower than $|\nabla u|^2$, e.g.

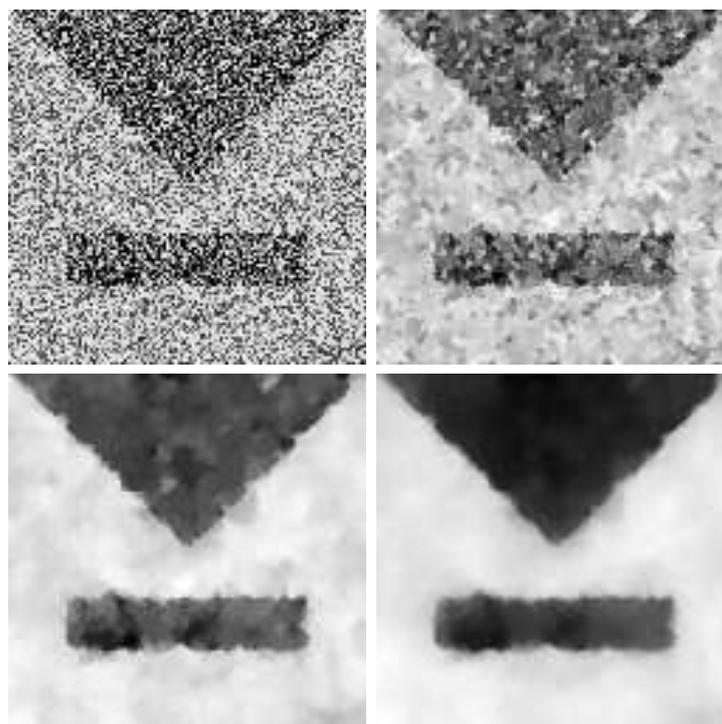
$$\Psi(|\nabla u|^2) := \lambda \sqrt{1 + |\nabla u|^2/\lambda^2}$$

- ◆ cf. robust, nonquadratic penalisation for M -smoothers (Lecture 12)

Discretisation

- ◆ yields nonlinear systems of equations (more cumbersome to solve)

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(a) **Top left:** Test image, 128×128 pixels. (b) **Top right:** Nonlinear variational method with $\lambda = 1$ and $\alpha = 20$. (c) **Bottom left:** $\alpha = 50$. (d) **Bottom right:** $\alpha = 100$. For improving visibility, an affine greyscale transformation to $[0, 255]$ has been performed. Author: J. Weickert (2000).

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Summary (1)



Summary

- ◆ Like their discrete counterparts, continuous variational methods minimise an energy with data and smoothness term. However, sums are replaced by integrals, and differences to neighbours by derivatives.
- ◆ The minimising function satisfies the Euler-Lagrange equation, a partial differential equation, supplemented with natural boundary conditions.
- ◆ Discretising the Euler–Lagrange equation leads to a
 - linear system of equations for a quadratic functional
 - nonlinear system for a nonquadratic functional.Such systems also arise directly from discrete variational methods.
- ◆ Nonquadratic variational methods can be useful for edge-preserving denoising.

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Summary (2)



Literature

- ◆ G. Aubert, P. Kornprobst: *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*. Second Edition, Springer, New York, 2006.
(one of the best books on variational methods for image analysis)
- ◆ A. M. Arthurs: *Calculus of Variations*. Routledge & Kegan Paul, London, 1975.
(booklet with all the essential informations on the calculus of variations)
- ◆ I. M. Gelfand, S. V. Fomin: *Calculus of Variations*. Dover, New York, 2000.
(well-readable and inexpensive book on the calculus of variations)
- ◆ R. Courant, D. Hilbert: *Methods of Mathematical Physics*. Vol. 1, Interscience, New York, 1953.
(excellent classical book with a large chapter on variational concepts; also in German)

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Assignment C4 (1)



Assignment C4 – Classroom Work

Problem 1 (Morphological Signal Processing)

Consider the following discrete 1-D signal

$$\mathbf{f} = (\dots, 0, 0, 3, 3, 3, 2, 1, 0, 0, 0, 1, 2, 1, 0, 0, 0, 2, 0, \dots)^T$$

which is extended by 0 on both sides. For some symmetric structuring element B_m of size $2m+1$ (from $-m$ until m) centered in 0 let the dilation and erosion be defined as:

$$(f \oplus B_m)_i := \max \{f_{i-j} \mid j = -m, \dots, m\},$$

$$(f \ominus B_m)_i := \min \{f_{i+j} \mid j = -m, \dots, m\}.$$

Use the signal above and compute dilation, erosion, opening and closing

- (a) for a structuring element of size 3
- (b) iterate it another time
- (c) for a structuring element of size 5

(This problem demonstrates some properties of morphological operations.)

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Assignment C4 (2)



Problem 2 (Nonlinear Diffusion)

In the explicit scheme for 1-D nonlinear diffusion filtering, the new value u_i^{l+1} at pixel i can be computed from the known values of the old time step k via

$$u_i^{k+1} = u_i^k + \frac{\tau}{h} \left(g_{i+1/2}^k \frac{u_{i+1}^k - u_i^k}{h} - g_{i-1/2}^k \frac{u_i^k - u_{i-1}^k}{h} \right)$$

where τ is the time step size, and $g_{i+1/2}^k$ approximates the diffusivity in the intermediate location $x_{i+1/2}$ at time level k . In the following let us assume that we have unit spatial grid size, i.e. $h = 1$.

- (a) How does the corresponding stencil look like ?
- (b) For the whole signal, this iteration step can be formulated as a single matrix–vector product

$$\mathbf{u}^{k+1} = Q(\mathbf{u}^k) \mathbf{u}^k.$$

Here, \mathbf{u}^k and \mathbf{u}^{k+1} are vectors of size N , and $Q(\mathbf{u}^k)$ is an $N \times N$ matrix.

Which structure has the matrix $Q(\mathbf{u}^k)$?

How it is related to the previous stencil ?

(This task connects the stencil notation for a single pixel to the overall explicit step.)

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Assignment T4 (1)



Assignment T4 – Theoretical Homework

Problem 1 (Continuous Nonquadratic Variational Methods)

(3+1+3 points)

Let $f(x)$ be a noisy continuous 1-D signal on the interval $[a, b]$. For discontinuity-preserving denoising, we seek a twice continuous differentiable function $u(x)$ that minimises the energy functional

$$E(u) := \int_a^b \left(\frac{1}{2} (u - f)^2 + \alpha \lambda^2 \sqrt{1 + u_x^2 / \lambda^2} \right) dx .$$

- Write it down the Euler-Lagrange equations of this energy functional.
- What is the meaning of the parameter λ ?
- Show that the Euler-Lagrange equation has a unique solution, and that this solution is indeed a minimiser (and not a maximiser) of the energy functional.

(This problem gives you insights on the calculus of variations.)

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Assignment T4 (2)



Problem 2 (Discrete Variational Methods)

(2+4 points)

- Write down a discrete analogue to the energy functional from Problem 1.
- Which nonlinear system of equations has to be satisfied necessarily in its minimum ?

(This problem connects continuous and discrete variational methods.)

Problem 3 (Wavelet Shrinkage)

(2+2+2 points)

Consider the following discrete 1-D signal

$$f = (3, 4, 1, -2, 0, 4, -2, -4)^T .$$

- Compute the discrete Haar wavelet transform of this signal.
- Perform hard wavelet shrinkage with a threshold parameter that removes three coefficients. Which value for the threshold parameter can achieve this ?
- Compute the wavelet backtransform.

(This problem shows the usefulness of wavelet transform for noise removal.)

Deadline for submission: Tuesday, January 8, 10 am (before the lecture).

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