Lecture 11:
Image Enhancement II:
Linear Filters

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1. Linear System Theory
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Linear System Theory

Linear System:

- A linear filter L satisfies the *superposition principle*:

$$L(\alpha f_1 + \beta f_2) \;=\; \alpha L f_1 + \beta L f_2$$

for all images f_1, f_2 and for all real numbers α, β.

Linear Shift Invariant (LSI) System:

- linear system, where the result of the filter depends only on the input signal, but not on the localisation of a signal structure:

$$L T_b f \;=\; T_b L f$$

for all translations T_b with $(T_b f)(x) := f(x - b)$.

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Impulse Response of an LSI System

- The *impulse response (Impulsantwort)* of an LSI filter L is the result of filtering a *discrete Dirac delta impulse*:

$$h = L\delta_0$$

where $\delta_0 = (\delta_{0,i})$ with the *Kronecker symbol*

$$\delta_{i,j} \;:=\; \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{else.} \end{cases}$$

- Since every discrete signal $f = (f_1, ..., f_N)^\top$ can be represented as linear combination of N unit impulses δ_1,...,δ_N, linearity and shift invariance imply

$$Lf \;=\; L\sum_{i=1}^{N} f_i\delta_i \;=\; \sum_{i=1}^{N} f_i L\delta_i \;=\; \sum_{i=1}^{N} f_i L T_i \delta_0 \;=\; \sum_{i=1}^{N} f_i T_i L\delta_0.$$

- This shows: *Any LSI system is completely characterised by its impulse response.*

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Example: Stock Market Price Averaged over the Last 200 Days

$$u_i \;:=\; \frac{1}{200}\sum_{k=0}^{199} f_{i-k}$$

This averaging can be represented as convolution (cf. Lecture 2):

$$u_i \;=\; \sum_{k=-\infty}^{\infty} f_{i-k}\, w_k \;=\; (f * w)_i$$

with the convolution mask

$$w_k := \begin{cases} \frac{1}{200} & k = 0, ..., 199, \\ 0 & \text{else.} \end{cases}$$

It is easy to check that this filter is linear and shift invariant.
Its impulse response $h = L\delta_0 = \delta_0 * w$ is given by the convolution mask:

$$h_i \;=\; \sum_{k=-\infty}^{\infty} \delta_{0,i-k}\, w_k \;=\; w_i.$$

Linear System Theory (4)

Repetition from Lecture 2: Convolution

- discrete convolution in 1-D:

$$(f * w)_i \; := \sum_{k=-\infty}^{\infty} f_{i-k}\, w_k$$

- discrete convolution in 2-D:

$$(f * w)_{i,j} \; := \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f_{i-k,\, j-l}\, w_{k,l}$$

Linear System Theory (5)

- continuous convolution in 1-D:

$$(f * w)(x) \; := \int_{-\infty}^{\infty} f(x - x')\, w(x')\, dx'$$

- continuous convolution in 2-D:

$$(f * w)(x, y) \; := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - x', y - y')\, w(x', y')\, dx'\, dy'.$$

Signals f with finite extension can e.g. be mirrored and extended periodically to an infinite image. Then it is sufficient to compute the convolution for one period.

Linear System Theory (6)

Important Properties of the Convolution

- **Linearity:**

$$(\alpha f + \beta g) * w \;=\; \alpha\,(f * w) + \beta\,(g * w) \qquad \forall\, \alpha, \beta \in \mathbb{R}.$$

- **Shift Invariance:**

$$T_b(f * w) = (T_b f) * w$$

  for all translations T_b.

- **Commutativity:**

$$f * w \;=\; w * f.$$

  Function and convolution kernel have equal rights.

- **Associativity:**

$$(f * v) * w \;=\; f * (v * w).$$

  Successive convolution with kernels v and w comes down to a single convolution with kernel $v * w$.

Linear System Theory (7)

- **Convolution Theorem of the Fourier Transform:**

$$\mathcal{F}[f * w] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[w]$$

  allows an efficient convolution if the kernels have a large extension

Importance of Convolutions in Linear System Theory

- A convolution is linear and shift invariant, i.e. it creates an LSI system.
- More importantly, it can be shown that even the reverse is true:
  An LSI system always performs a convolution !
- The convolution mask is given by the impulse response.

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Importance of the Fourier Transform in Linear System Theory

- Since convolutions in the spatial domain come down to multiplications in the Fourier domain, Fourier analysis plays a fundamental role for understanding LSI filters
- For large convolution kernels, it is more efficient to perform the computation in the Fourier domain.
- LSI filters are often studied in the Fourier domain, in order to understand their frequency behaviour.
- Often one even starts designing LSI filters in the Fourier domain, and transforms them back to the spatial domain afterwards.

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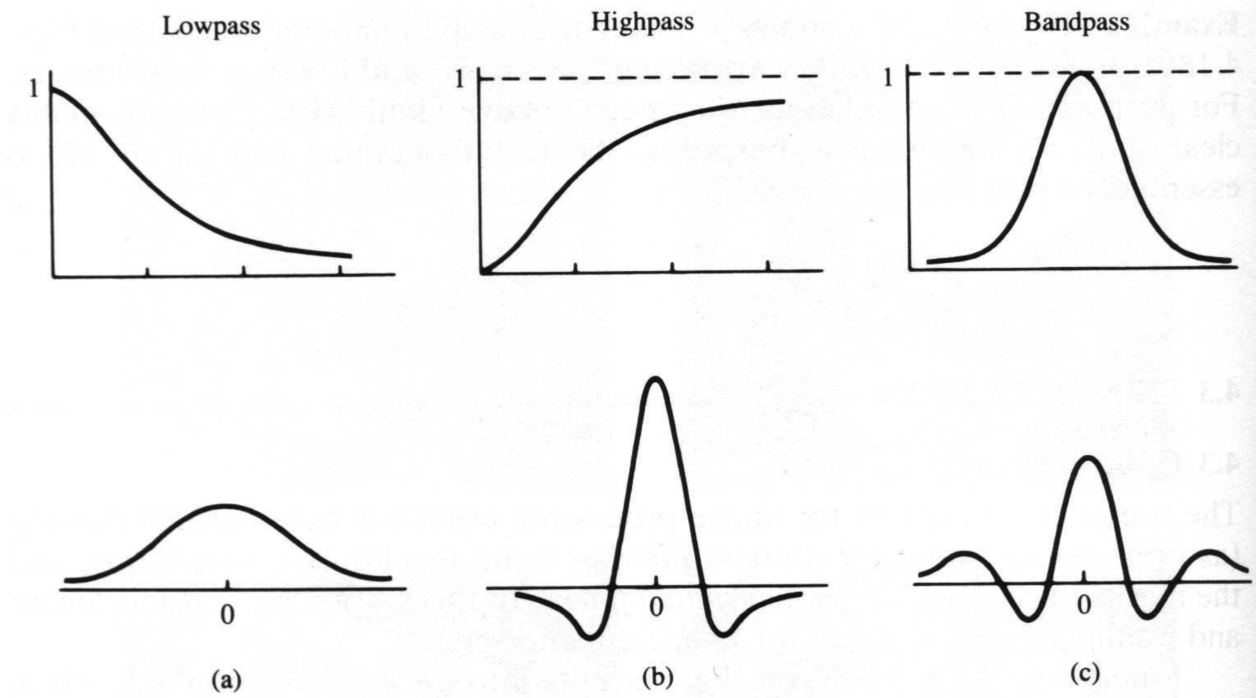
Basic Types of LSI Filters

- *lowpass filters:* low frequencies are less attenuated than high ones
- *highpass filters:* high frequencies are less attenuated than low ones
- *bandpass filters:* structures within a specific frequency band are hardly attenuated



(a) Top: Basic types of convolution kernels in the Fourier domain. **(b) Bottom:** Corresponding kernels in the spatial domain. Authors: R. C. Gonzalez, R. E. Woods (1992).

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Lowpass Filters

Goals

- smooth an image by eliminating noise and unimportant small-scale details
- design in the spatial domain: convolution with averaging masks
- design in the Fourier domain: attenuate high frequencies

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Lowpass Filters (2)

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Design in the Spatial Domain: Box Filters

- convolution mask of size $(2m+1) \times (2m+1)$ with weights $\frac{1}{(2m+1)^2}$
- can also be implemented efficiently for large masks in the spatial domain:
  - filter is separable
  - shifting the 1-D mask by one pixel to the right removes one grey value at the left end and adds one at the right end
  - total complexity is linear and independent of the mask size:
    1 addition, 1 subtraction, 1 multiplication per pixel (in 1-D)
- results do not often look too convincing:
  - not rotationally invariant: prefers horizontal and vertical structures
  - continuous FT of a box function is a sinc function (Lecture 3)
  - not satisfactory in the frequency domain:
    multiple maxima, attenuation of high freqencies only with $1/|u|$

Lowpass Filters (3)

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Optimality in the Frequency Domain: The Ideal Lowpass

- All frequency components (u, v) with $u^2 + v^2 > T^2$ are set to 0.
- also not satisfactory:
  sinc-like rotationally invariant function in the spatial domain with visible ringing artifacts

Lowpass Filters (4)

Gaussian Convolution Kernels

- m-dimensional Gaussian:

$$K_\sigma(x) := \frac{1}{(2\pi\sigma^2)^{m/2}} \exp\left(-\frac{|x|^2}{2\sigma^2}\right).$$

  "width" σ is called *standard deviation*, σ^2 is the *variance*.

- creates Gaussian with reciprocal variance in Fourier domain (Lecture 3)
- good compromise: one maximum in both spatial and frequency domain
- the only convolution kernel that is both separable and rotationally invariant
- Iterated Gaussian convolution creates a new Gaussian where the variances sum up.
- Gaussian convolution can be implemented efficiently in numerous ways.

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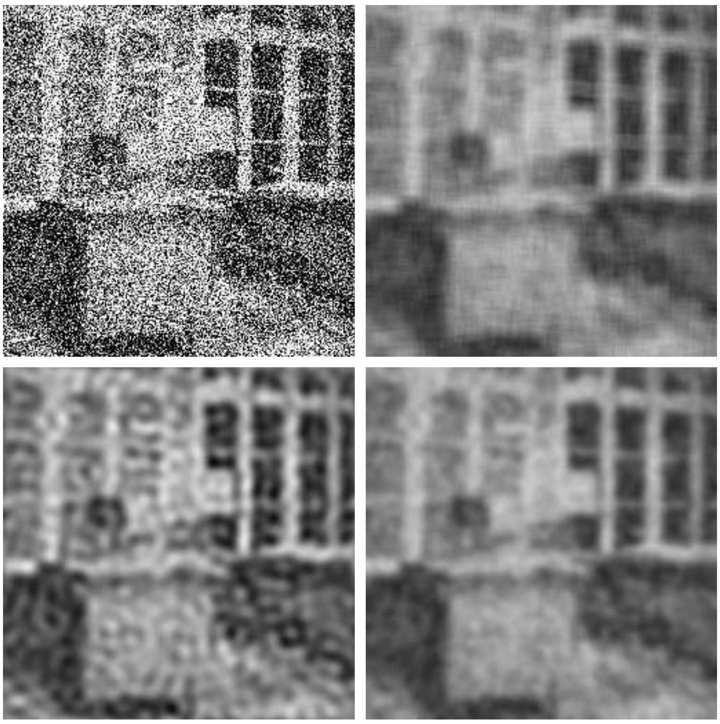
Lowpass Filters (5)



(a) Top left: Noisy original image. **(b) Top right:** Filtering with a 11×11 box filter creates horizontal and verticale artifacts. **(c) Bottom left:** The ideal lowpass with $T^2 = 500$ creates ringing artifacts. **(d) Bottom right:** Smoothing with a Gaussian with $\sigma = 3$ gives much better results. Author: J. Weickert (2000).

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Approximation Possibility 1: Sampling of the Gaussian in the Spatial Domain

- exploit separability and symmetry in order to achieve high efficiency
- restrict sampling to interval $[-k\sigma, k\sigma]$ (high accuracy for $k \geq 3$)
- renormalise sume of coefficients to 1
- Advantage: simple and flexible (σ can be tuned continuously)
- Disadvantage: computational complexity increases with σ
- good for small values of σ

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Approximation Possibility 2: Multiplication in the Fourier Domain

- cf. Lecture 4:
  - use FFT to transform the image into the Fourier domain
  - multiply with the Fourier transform of the Gaussian (a Gaussian with inverse variance)
  - use FFT for backtransformation
- Advantages:
  - almost linear complexity: $\mathcal{O}(N^2 \log N)$ for an $N \times N$ image
  - computational complexity does not increase with σ
- Disadvantages:
  - wraparound errors
  - standard FFT requires image sizes of powers of 2
- good for large values of σ

Lowpass Filters (8)

Approximation Possibility 3: Binomial Kernels

Normalisation	Filter Coefficients	Variance σ^2
1	1	0
1/2	1 1	1/4
1/4	1 2 1	1/2
1/8	1 3 3 1	3/4
1/16	1 4 6 4 1	1
1/32	1 5 10 10 5 1	5/4
1/64	1 6 15 20 15 6 1	3/2
1/128	1 7 21 35 35 21 7 1	7/4
1/256	1 8 28 56 70 56 28 8 1	2

- Limit theorem of Moivre–Laplace: Binomial kernels approximate Gaussians
- exploit separability and symmetry
- iterated binomial kernels create binomial kernels
- Advantage: possible in integer arithmetics
- Disadvantages:
  - computational complexity increases with σ
  - σ cannot be tuned continuously

Lowpass Filters (9)

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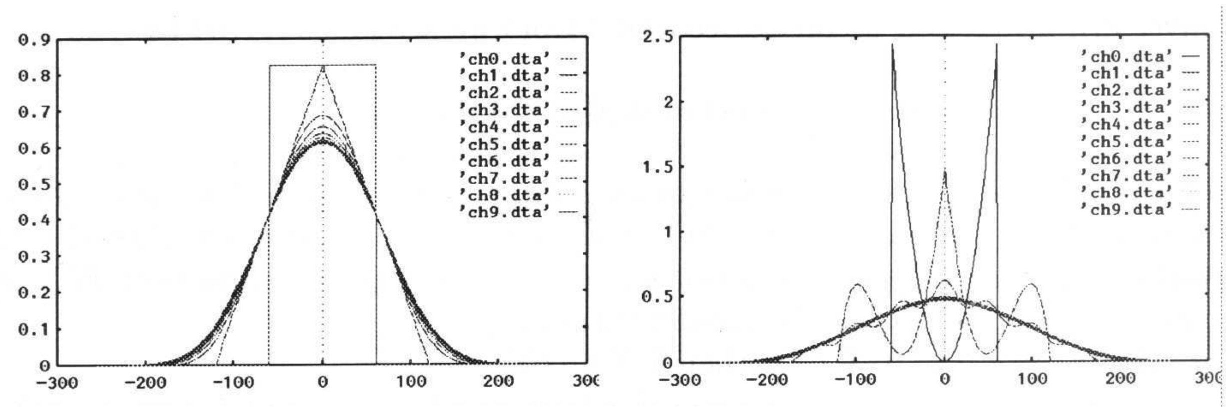
Approximation Possibility 4: Iterated Box Filters



Left: Iterated box filters approximate a Gaussian. **Right:** Iterating a more complicated filter also approximates a Gaussian. Authors: F. Guichard, J.-M. Morel (1999).

- Central limit theorem of statistics: Iterated averaging kernels converge to Gaussians
- Iterating a box filter three times is already a good approximation.
- Advantage: linear complexity, independent of σ
- Disadvantage: σ cannot be tuned continuously

Highpass Filters

Goals:

- remove low-frequent background perturbations
- sharpen blurry image structures

Remarks:

- An important class of highpass filters consists of derivative filters for detecting edges (Lecture 18).
- While lowpass filters act stabilising, highpass filters that enhance high frequencies usually act destabilising.

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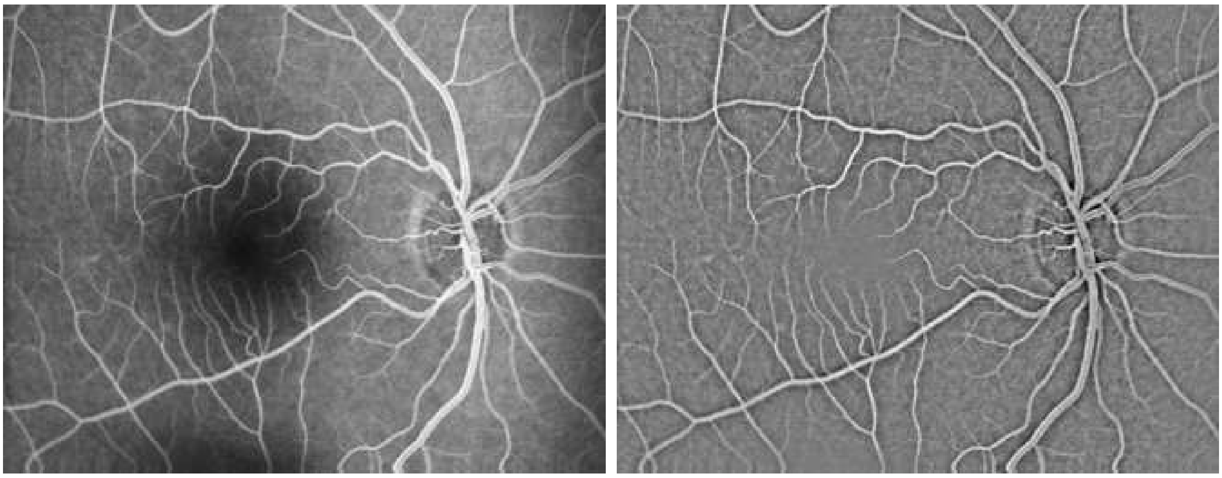
Design in the Spatial Domain

- Example: Highpass filter as difference between identity and lowpass.
- A 3×3 box filter, for instance, creates the highpass stencil

$$\begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ \hline \end{array} - \begin{array}{|c|c|c|}\hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \hline -\frac{1}{9} & \frac{8}{9} & -\frac{1}{9} \\ \hline -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ \hline \end{array}$$

- Displaying the filtered image often requires an affine rescaling of the grey values (cf. Lecture 10):
  - For many of these filters, the average grey value becomes 0 (e.g. if the lowpass filter preserves the average grey value).
  - Thus, negative values are common.



(a) Left Vessel structure of the background of the eye. **(b) Right:** Elimination of low-frequent background structures by subtracting a Gaussian-smoothed version from the original image. The greyscale range [-94,94] has been rescaled to $[0, 255]$ by an affine rescaling. Author: J. Weickert (2002).

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Bandpass Filters

- not so important for image enhancement, but useful for extracting interesting image structures on certain scales
- Example: assessing the cloudiness of fabrics (Lecture 5)
- can be created by subtracting two lowpass filters
- If the lowpass filters are Gaussians, the resulting bandpass is called *DoG (difference of Gaussians)*.

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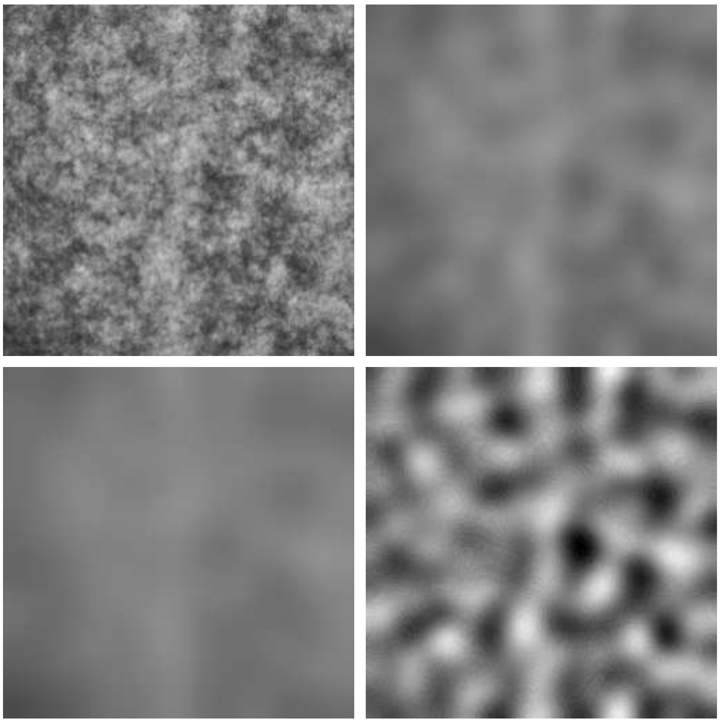
Bandpass Filters (2)



(a) Top left: Fabric, 257×257 pixels. **(b) Top right:** After lowpass filtering with a Gaussian with $\sigma = 10$. **(c) Bottom left:** lowpass filtering with $\sigma = 15$. **(d) Bottom right:** Subtracting (b) and (c) gives a bandpass filter that visualises cloudiness on a certain scale. The greyscale range has been affinely rescaled from $[-13, 13]$ to $[0, 255]$. Author: J. Weickert (2002).

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Summary (1)

Summary

- Linear shift invariant (LSI) filters are fully characterised by their impulse response.
- can always be represented as convolutions
- impulse response: given by convolution mask
- The Fourier transform is very important for designing LSI filters.
- Lowpass filters are useful for smoothing data.
- most important example: Gaussian convolution
- Highpass filters eliminate low-frequent perturbations and/or sharpen image structures.
- Bandpass filters are mainly used for extracting features at certain scales.

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Literature

- K. R. Castleman: *Digital Image Processing*. Prentice Hall, Upper Saddle River, 1996.
  (a text book that focuses on LSI filters)
- R. C. Gonzalez, R. E. Woods: *Digital Image Processing*. Pearson, Upper Saddle River, Third Edition, 2008.
  (see in particular Chapter 4)
- A. V. Oppenheim, R. W. Schafer, J. R. Buck: *Discrete-Time Signal Processing*. Prentice Hall, Englewood Cliffs, Second Edition, 1999.
  (Signal processing books usually provide an exhaustive treatment of LSI filters. This volume is one of the classical text books on signal processing.)