Lecture 5:
Image Transformations III:
Image Pyramids

Contents

Gaussian Pyramids (Gauß–Pyramiden)

Goal:

- image representation on multiple resolution levels
- going to the next coarser resolution should reduce the image size in each dimension by a factor of (approximately) 2.

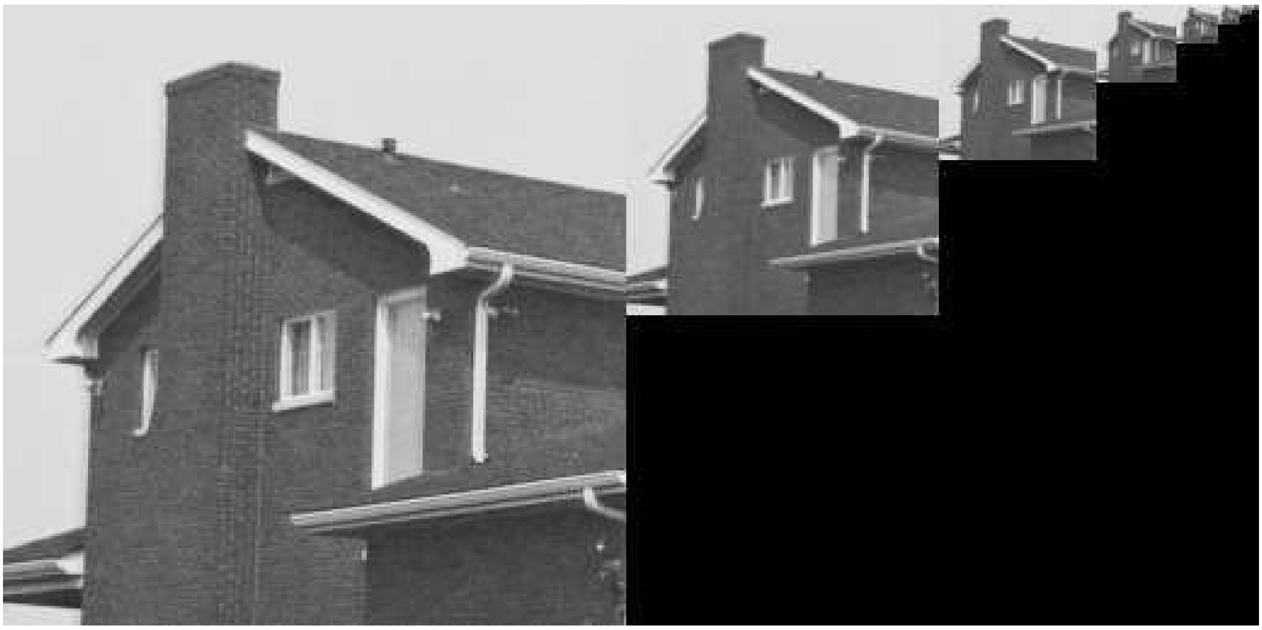
How to Reduce a Signal to Half its Size

- Sampling theorem (Lecture 3):
  Attenuate high frequences in order to avoid aliasing effects.
- Thus, first smooth the signal by averaging, then sample with half the frequency.
- simple and widely use smoothing mask in 1-D:

$$\text{binomial kernel} \quad \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$$

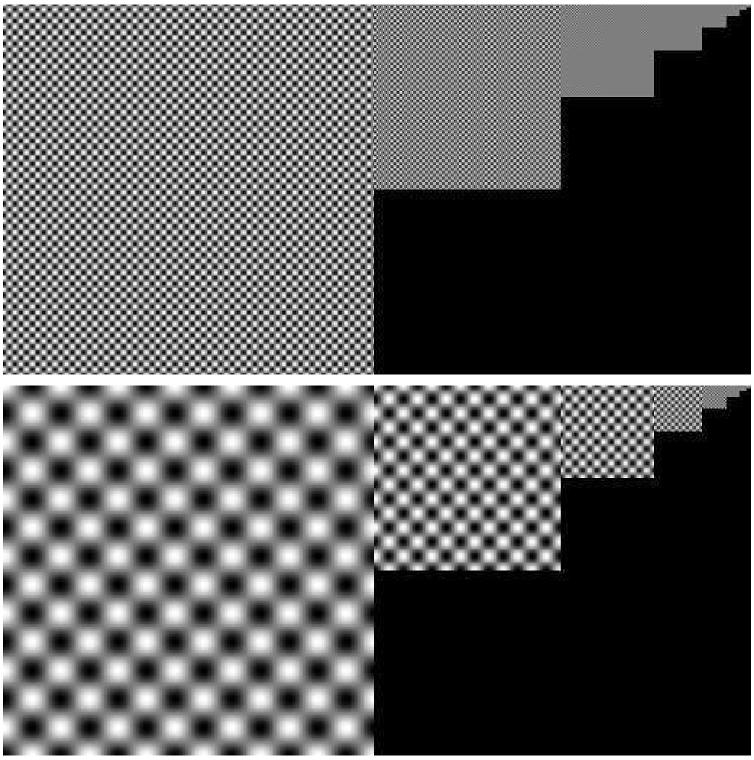
- in m dimensions: apply this mask subsequently along all m axes (separability)

Gaussian Pyramids (2)



Gaussian pyramid of an image of size 257×257. Author: J. Weickert (2000).

Gaussian Pyramids (3)



(a) Top: Gaussian pyramid of a test image with high frequencies. **(b) Bottom:** Gaussian pyramid of a test image with low frequencies. This shows that the Gaussian pyramid is a lowpass filter: Lower frequencies may pass it better than higher ones. Author: J. Weickert (2000).

Formal Description of the 1-D Gaussian Pyramid

- consider 1-D signal $u = (u_0, ..., u_{2^N})^\top$
- signal at level k has $2^k + 1$ values for $k \geq 1$, and 1 value for $k = 0$.
- Define *restriction operators* from level k to level $k-1$ as multiplication with

$$R_k^{k-1} \; := \; \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \qquad (k \geq 2),$$

$$R_1^0 \; := \; \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

(special treatment of boundaries will be discussed later)

- For $k \geq 2$ the restriction operator R_k^{k-1} is a $(2^{k-1}+1) \times (2^k + 1)$ matrix.
- The *Gaussian pyramid* $\{v^N, ..., v^0\}$ of u is defined as

$$\begin{aligned} v^N &:= u, \\ v^{k-1} &:= R_k^{k-1} v^k \qquad (k = N, ..., 1). \end{aligned}$$

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Complexity Aspects

- The Gaussian pyramid requires hardly more disk space that the original image: Because of $\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}$ for $|q| < 1$, the (approximate) factors are

$$\text{in 1-D:} \quad 1 + \frac{1}{2} + \frac{1}{4} + \ldots = \frac{1}{1-\frac{1}{2}} = 2$$

$$\text{in 2-D:} \quad 1 + \frac{1}{4} + \frac{1}{16} + \ldots = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

$$\text{in 3-D:} \quad 1 + \frac{1}{8} + \frac{1}{64} + \ldots = \frac{1}{1-\frac{1}{8}} = \frac{8}{7}$$

- The pyramid decomposition has optimal complexity: linear!
- There even exist hardware realisations of image pyramids.
- Unfortunately, pyramids are not invariant under translations.

Coarse-to-Fine Strategies

- Many image processing methods are iterative.
  A good initialisation leads to fast convergence.
- widely used strategy to obtain good initialisations:
  - apply iterative method at a coarse level of the Gaussian pyramid first
  - allows to obtain quickly a coarse scale approximation to the fine scale solution
  - interpolate it and use it as initialisation at the next finer scale
- additional advantages:
  - coarse scales are robust under noise and small scale perturbations
  - also useful in optimisation methods for avoiding getting trapped in irrelevant local minima
- related technique in numerical analysis: *multigrid methods*
  (allow e.g. to solve linear systems of equations in linear complexity)

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Laplacian Pyramids (Laplace–Pyramiden)

Goals:

- decompose an image in its *spatial domain (!)* in different frequency bands
- alternative to Fourier analysis that requires to work in the *frequency domain*

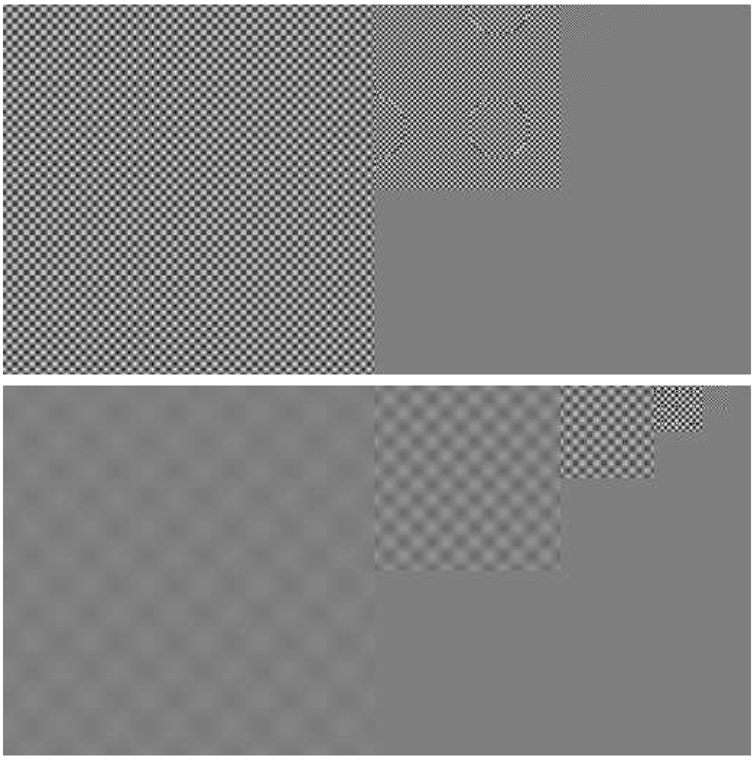
Basic Idea:

- downsampling in the Gaussian pyramid attenuates higher frequencies while lower frequencies may pass *(lowpass filter)*.
- difference of subsequent levels single out certain frequency bands *(bandpass filter)*.
- For subtracting images of equal size, interpolate the downsampled image.
  Simple and widely used interpolation strategy: linear interpolation (separable)
- This pyramid of difference images is called *Laplacian pyramid*.

Laplacian Pyramids (2)

Laplacian pyramid of an image of size 257×257. Author: J. Weickert (2000).

Laplacian Pyramids (3)



(a) Top: Laplacian pyramid of a test image with high frequencies. **(b) Bottom:** Laplacian pyramid of a test image with low frequencies. One observes that the Laplacian pyramid is a bandpass filter. Author: J. Weickert (2000).

Formal Description of the 1-D Laplacian Pyramid

- Given: signal $u = (u_0, ..., u_{2^N})^\top$ with Gaussian pyramid $\{v^N, ..., v^0\}$
- define *interpolation operator (prolongation operator)* from level k to level $k+1$ as multiplication with the matrix

$$P_k^{k+1} := \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix} \qquad (k \geq 1),$$

$$P_0^1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

- P_k^{k+1} is a $(2^{k+1}+1) \times (2^k+1)$ matrix for $k \geq 1$.
- The *Laplacian pyramid* $\{w^N, ..., w^0\}$ of u is computed as

$$\begin{aligned} w^k &:= v^k - P_{k-1}^k v^{k-1} \qquad (k = N, ..., 1), \\ w^0 &:= v^0. \end{aligned}$$

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19

Remarks

- Interpolation and restriction operators are designed in such a way that v^k and its smoothed variant $P^k_{k-1} R^{k-1}_k v^k$ have the same average grey value.
- This expains the specific boundary treatment in the restriction (weights $\frac{2}{3}$ and $\frac{1}{3}$).
- Each level of the Laplacian pyramid has average grey value 0, except for w^0.
- The Laplacian pyramid allows to reconstruct the original image via the Gaussian pyramid:

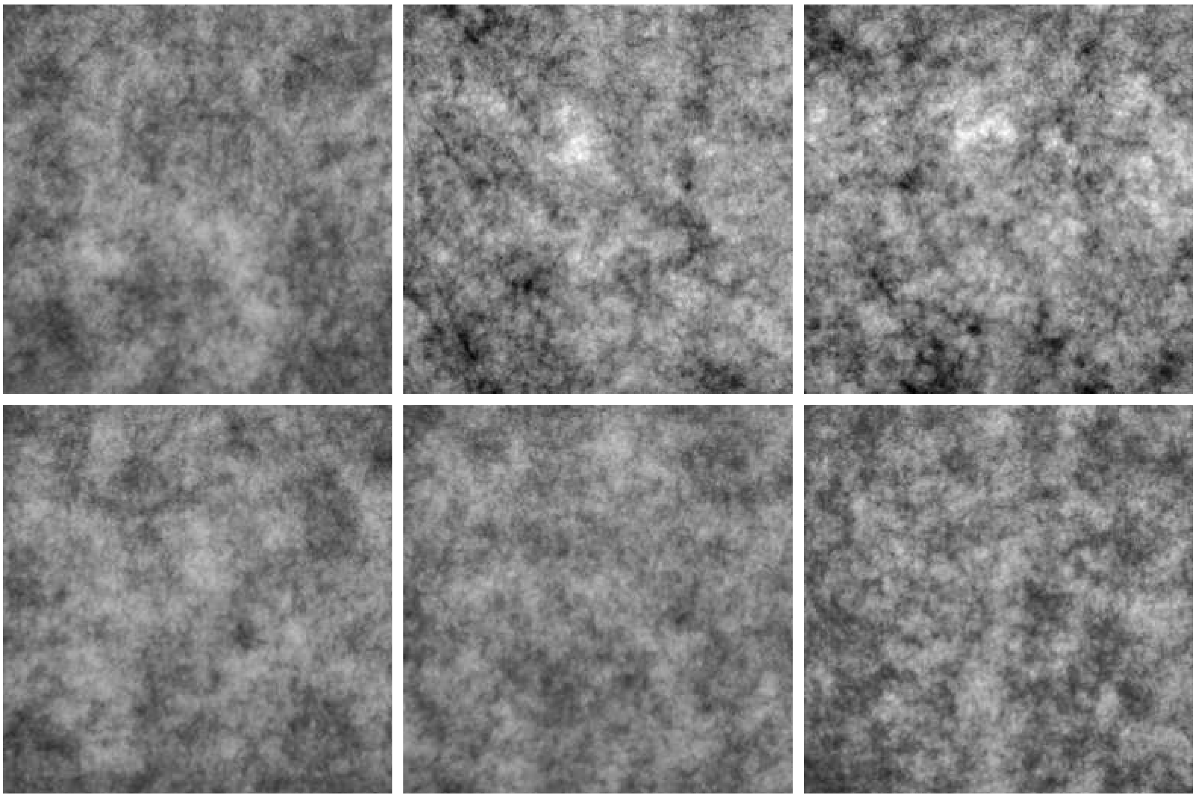
$$\begin{aligned} v^0 &:= w^0, \\ v^k &:= w^k + P^k_{k-1} v^{k-1} \qquad (k = 1, ..., N), \\ u &:= v^N. \end{aligned}$$

- Although the Laplacian pyramid contains more data than the original image, it has even been used for image compression:
  High-frequent components can be quantised more coarsely without visible degradations.

An Application of Laplacian Pyramids: Quality Control of Nonwoven Fabrics (Spinnvlies)

- cloudiness is an essential quality parameter for nonwoven fabrics
- company wants to automise quality control
- cloudiness is a scale phenomenon:
  small clouds are less important that larger ones
- bandpass decomposition of the fabric images using a Laplacian pyramid
- quality parameter: weighted average of the grey value variances at the different levels
- weights have been found through test series
- reliability of the system as good as a human expert
- Laplacian pyramids are very fast:
  online quality control of the entire production line on a single PC

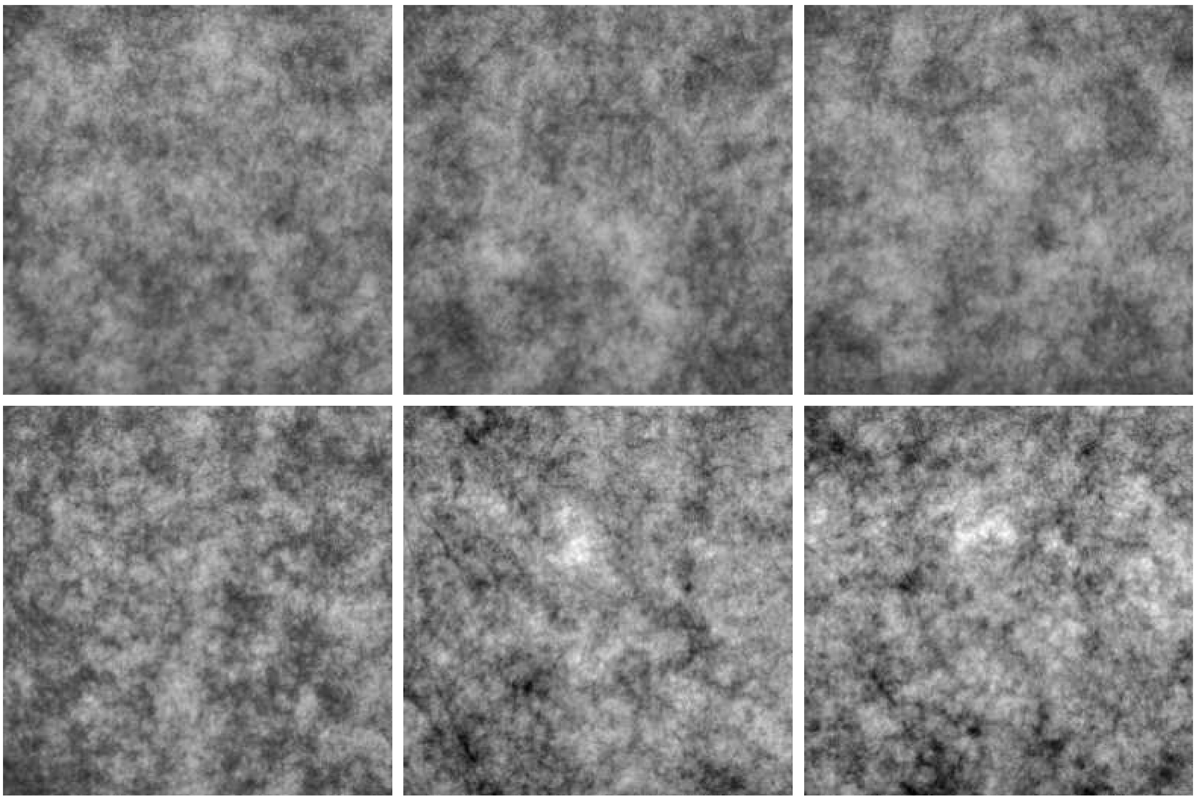
An Application of Laplacian Pyramids (2)



Six characteristic fabric images. The goal is to order them according to their cloudiness. Author: J. Weickert (1999).

An Application of Laplacian Pyramids (3)

1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19



From left to right and from top to bottom: Increasing cloudiness according to the automised control system. Author: J. Weickert (1999).

An Application of Laplacian Pyramids (4)

Result of the farbric classification of the six images. c is the quality parameter of the system, and r is the average judgement (with standard deviation) of 32 humans. This shows that the system is as reliable as these humans.

image number	#5	#1	#4	#6	#2	#3
c	48.32	51.79	52.26	66.59	85.26	100.63
r	1.50 $\pm$ 0.56	2.28 $\pm$ 0.51	2.28 $\pm$ 0.45	4.38 $\pm$ 0.54	5.03 $\pm$ 0.17	5.53 $\pm$ 0.50

Summary (1)

Summary

- Pyramids are image representations in the spatial domain.
- Gaussian pyramid is a lowpass filter,
  Laplacian pyramid a bandpass decomposition
- Pyramid decompositions have linear complexity.
- Gaussian pyramidas can speed up algorithms,
  Laplacian pyramids give a frequency decomposition in the spatial domain.
- Pyramids require a little bit more disk space than the original image.
- not invariant under translations
- A correct boundary treatment is particularly important at coarse levels.

Literature

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  (gentle introduction to multigrid methods for solving linear systems of equations)
- J. Weickert: A real-time algorithm for assessing inhomogeneities in fabrics. *Real-Time Imaging*, Vol. 5, No. 1, pp. 15–22, Febr. 1999.
  www.mia.uni-saarland.de/weickert/Papers/defects.ps.gz
  (Laplacian pyramids for quality assessment of fabrics)