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Lecture 3:
Image Transformations I:
Continuous Fourier Transform

Contents

Continuous Fourier Transform in 1-D (1)

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Joseph Fourier (1768–1830) did not only discover the so-called Fourier transform, he also introduced the diffusion equation, made archeological discoveries in Egypt, and acted as a prefect in Grenoble. Source: `www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Fourier.html`.

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Continuous Fourier Transform in 1-D

Goal:

- decompose a signal into its frequency components
- indispensable tool for understanding linear filters
- algorithmically important: expensive convolution operations become inexpensive

Definition:

The *Fourier transform* of a 1-D function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$\hat{f}(u) \;:=\; \mathcal{F}[f](u) \;:=\; \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx$$

where i is the complex number with $i^2 = -1$.

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Interpretation:

- $\hat{f}(u)$ is the coefficient of f with respect to the inner product (Skalarprodukt)

  $$\langle f, g \rangle := \int_{-\infty}^{\infty} f(x)\, \bar{g}(x)\, dx$$

  and the basis function $g_u(x) := e^{i2\pi ux}$ with some “frequency” $u \in \mathbb{R}$.
  ($\bar{z} := x - iy$ denotes the complex conjugate of $z = x + iy$.)

- $\hat{f}(u)$ is complex-valued.

- Because of Euler’s formula $e^{i2\pi ux} = \cos(2\pi ux) + i \sin(2\pi ux)$ one can regard
  - the real part $\mathrm{Re}(\hat{f}(u))$ as the component of f with respect to $\cos(2\pi ux)$
  - the imaginary part $\mathrm{Im}(\hat{f}(u))$ as the component of f with respect to $\sin(2\pi ux)$

- Thus, $\{\hat{f}(u) \,|\, u \in \mathbb{R}\}$ is a representation of the function $\{f(x) \,|\, x \in \mathbb{R}\}$ in the *frequency domain (Fourier domain)*.

Continuous Fourier Transform in 1-D (4)

Definition:

The *inverse 1-D Fourier transform* of $\hat{f}(u)$ is defined as

$$f(x) \;=\; \mathcal{F}^{-1}[\hat{f}](x) \;:=\; \int_{-\infty}^{\infty} \hat{f}(u)\, e^{i2\pi ux}\, du$$

Interpretation:

- The Fourier transform decomposes a signal $\{f(x) \,|\, x \in \mathbb{R}\}$ into its frequency components $\{\hat{f}(u) \,|\, u \in \mathbb{R}\}$. The frequency component with respect to a frequency u is obtained by projecting $f(x)$ on the basic function $g_u(x) = e^{i2\pi ux}$:

$$\hat{f}(u) \;=\; \langle f, g_u \rangle.$$

- The inverse Fourier transform synthesises the signal $\{f(x) \,|\, x \in \mathbb{R}\}$ from its frequencies $\{\hat{f}(u) \,|\, u \in \mathbb{R}\}$ w.r.t. the basis functions $\{g_u(x) = e^{i2\pi ux} \,|\, u \in \mathbb{R}\}$:

$$f(x) \;=\; \int_{-\infty}^{\infty} \langle f, g_u \rangle\, g_u\, du.$$

Continuous Fourier Transform in 1-D (5)

Remarks

- Euler's formula $e^{i\phi} = \cos\phi + i\sin\phi$ gives the useful relations

$$\cos\phi \;=\; \frac{e^{i\phi} + e^{-i\phi}}{2}, \qquad \sin\phi \;=\; \frac{e^{i\phi} - e^{-i\phi}}{2i}.$$

- One defines the *Fourier spectrum (Fourierspektrum)* as

$$|\hat{f}(u)| = \sqrt{\mathrm{Re}^2(\hat{f}(u)) + \mathrm{Im}^2(\hat{f}(u))}$$

  and the *phase angle (Phasenwinkel)* as

$$\phi(\hat{f}(u)) = \arctan\left(\frac{\mathrm{Im}(\hat{f}(u))}{\mathrm{Re}(\hat{f}(u))}\right).$$

- Often one is interested only in the Fourier spectrum $|\hat{f}(u)|$ or the *power spectrum (Powerspektrum)* $|\hat{f}(u)|^2$.
  They describe the importance of the frequency u within the signal f.

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Continuous Fourier Transform in 1-D (6)

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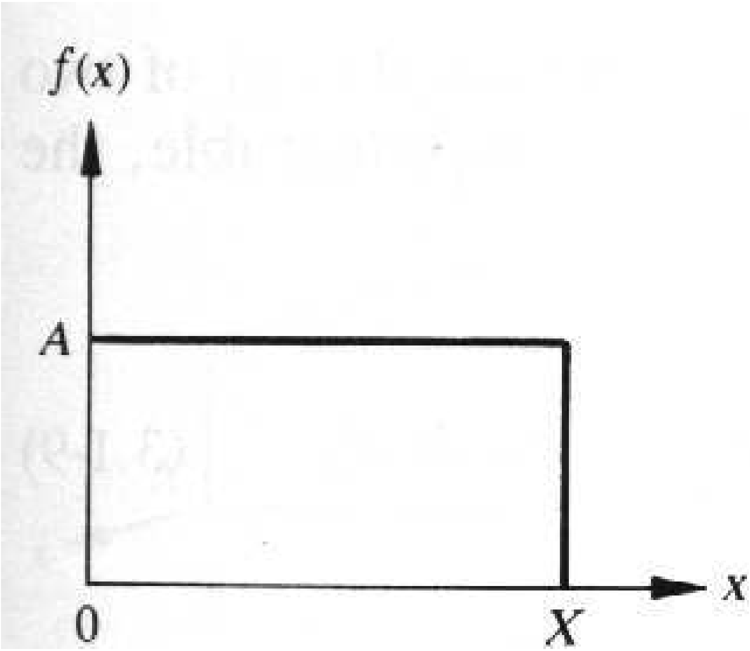
Example: Fourier Transform of a Box Function



A box function. Source: R. C. Gonzalez, R. E. Woods (1992).

Continuous Fourier Transform in 1-D (7)

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The Fourier transform of this box function is given by

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(x)\, e^{-i2\pi ux}\, dx \\
&= \int_{0}^{X} A\, e^{-i2\pi ux}\, dx \\
&= \frac{-A}{i2\pi u} \left[e^{-i2\pi ux}\right]_0^X \\
&= \frac{-A}{i2\pi u} \left(e^{-i2\pi uX} - 1\right) \\
&= \frac{-A}{i2\pi u} e^{-i\pi uX} \left(e^{-i\pi uX} - e^{i\pi uX}\right) \\
&= \frac{-A}{i2\pi u} e^{-i\pi uX} \left(-2i \sin(\pi uX)\right) \\
&= \frac{A}{\pi u} e^{-i\pi uX} \sin(\pi uX)
\end{aligned}$$

Continuous Fourier Transform in 1-D (8)

With $|e^{-i\pi uX}| = 1$ we obtain the Fourier spectrum

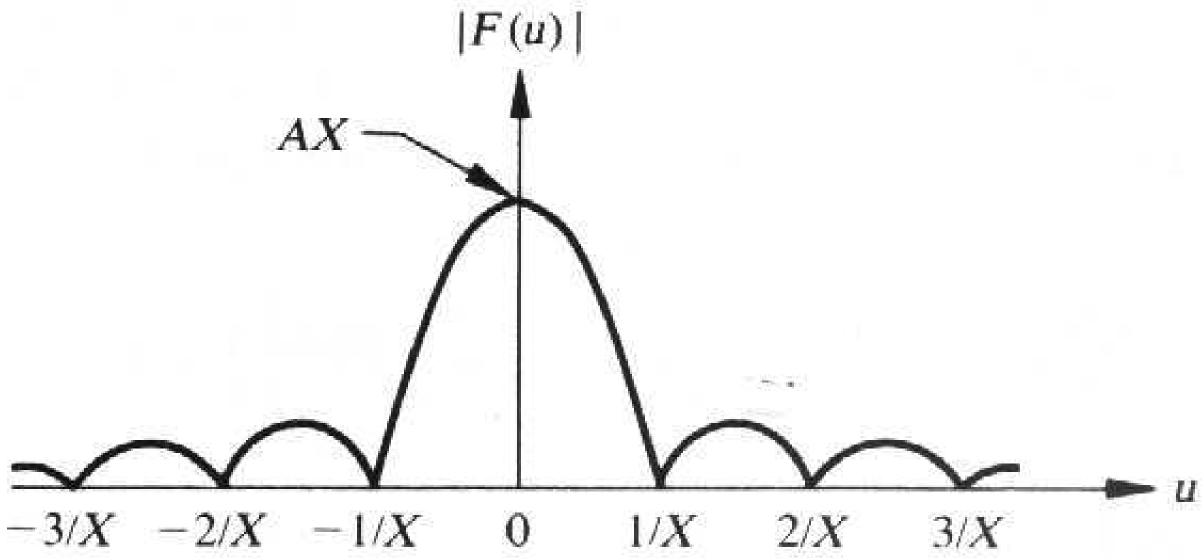
$$\begin{aligned} |\hat{f}(u)| &= \left|\frac{A}{\pi u}\right| \, |\sin(\pi u X)| \\ &= AX \left|\frac{\sin(\pi u X)}{\pi u X}\right| \\ &= AX \, |\text{sinc}(\pi u X)| \end{aligned}$$

with the so-called *sinc function*

$$\text{sinc}(x) := \frac{\sin(x)}{x}.$$

Remark:
$f(x)$ has finite extent in the spatial domain, but:
$\hat{f}(u)$ has infinite extent in the frequency domain.

Continuous Fourier Transform in 1-D (9)



Fourier spectrum $|\hat{f}(u)|$ of the box function. Source: R. C. Gonzalez, R. E. Woods (1992).

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Continuous Fourier Transform in 2-D

Definition:

- The *Fourier transform (FT)* of a 2-D function $f(x, y)$ is defined as

$$\hat{f}(u, v) \;:=\; \mathcal{F}[f](u, v) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$

- The *inverse 2-D Fourier transform* is given by

$$f(x, y) \;=\; \mathcal{F}^{-1}[\hat{f}](x, y) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(u, v)\, e^{i2\pi(ux+vy)}\, du\, dv.$$

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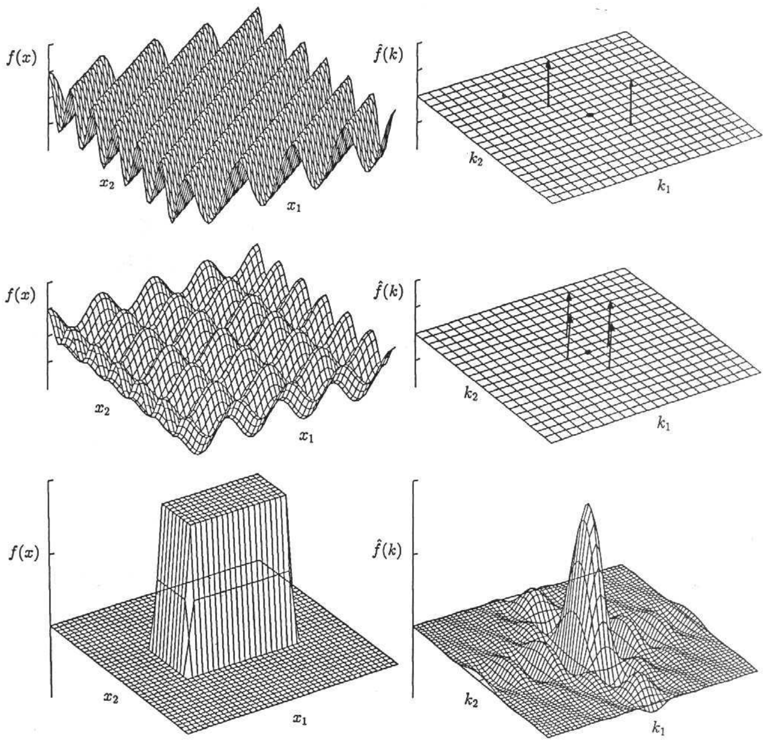
What About Higher Dimensions ?

- In higher dimensions one proceeds in an analogue way.
- Because of

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, e^{-i2\pi(ux+vy)}\, dx\, dy$$
$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x, y)\, e^{-i2\pi ux}\, dx\right) e^{-i2\pi vy}\, dy$$

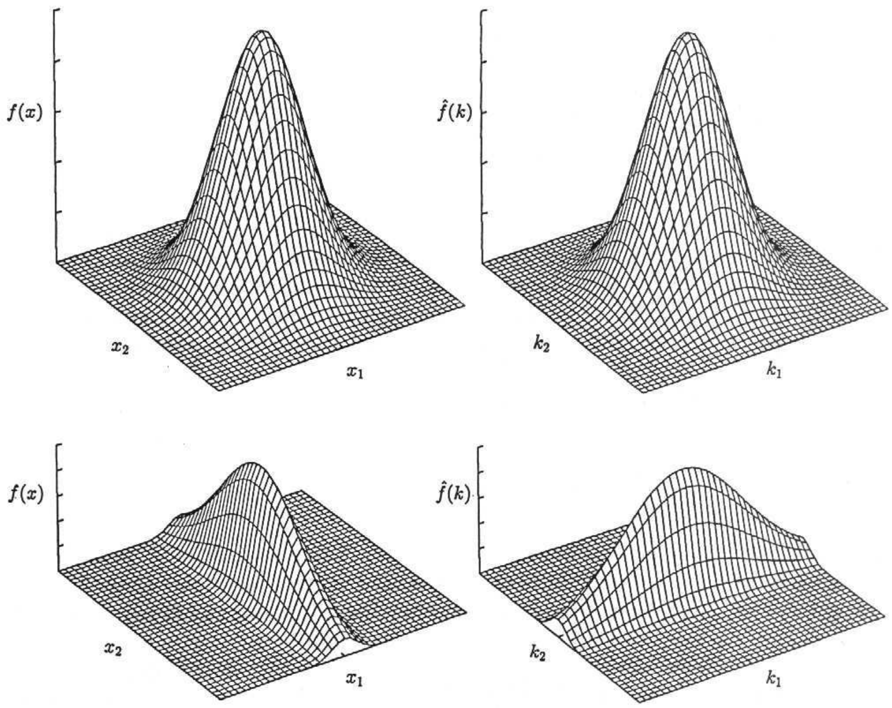
if follows that the Fourier transform is *separable:*
An m-dimensional transform can be computed via a sequence of m one-dimensional transforms.

Continuous Fourier Transform in 2-D (3)



Fourier spectra of some 2-D functions. Author: B. Jähne (1991).

Continuous Fourier Transform in 2-D (4)



Further 2-D Fourier spectra. Author: B. Jähne (1991).

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Properties of the Continuous Fourier Transform

- **Linearity**

$$\mathcal{F}[af+bg] \;=\; a\mathcal{F}[f]+b\mathcal{F}[g] \qquad \forall\, a,b\in\mathbb{R}.$$

The superposition principle holds.

- **Similarity Theorem**

$$\mathcal{F}[f(ax,by)](u,v) \;=\; \frac{1}{|ab|}\mathcal{F}[f]\left(\frac{u}{a},\frac{v}{b}\right) \qquad \forall\, a,b\in\mathbb{R}\setminus\{0\}.$$

Elongation in the spatial domain gives shortening in the Fourier domain:
Both domains are reciprocal.

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- **Differentiation**

$$\mathcal{F}\left[\frac{\partial^{n+m} f}{\partial x^n \partial y^m}\right] \;=\; (i2\pi u)^n\,(i2\pi v)^m\,\mathcal{F}[f](u,v).$$

Differentiation in the spatial domain gives multiplication with the frequency in the Fourier domain. Thus, high frequent components (e.g. noise) are amplified!

- **Shift Theorem**

$$\mathcal{F}[f(x-x_0,\, y-y_0)](u,v) \;=\; e^{-i2\pi(ux_0+vy_0)}\,\mathcal{F}[f](u,v)$$

Shift in space domain rotates phase angle in Fourier domain. The Fourier *spectrum*, however, is not affected! In this sense the FT is shift invariant.

- **Rotation Invariance**

If the image is rotated, its FT is rotated by the same angle.

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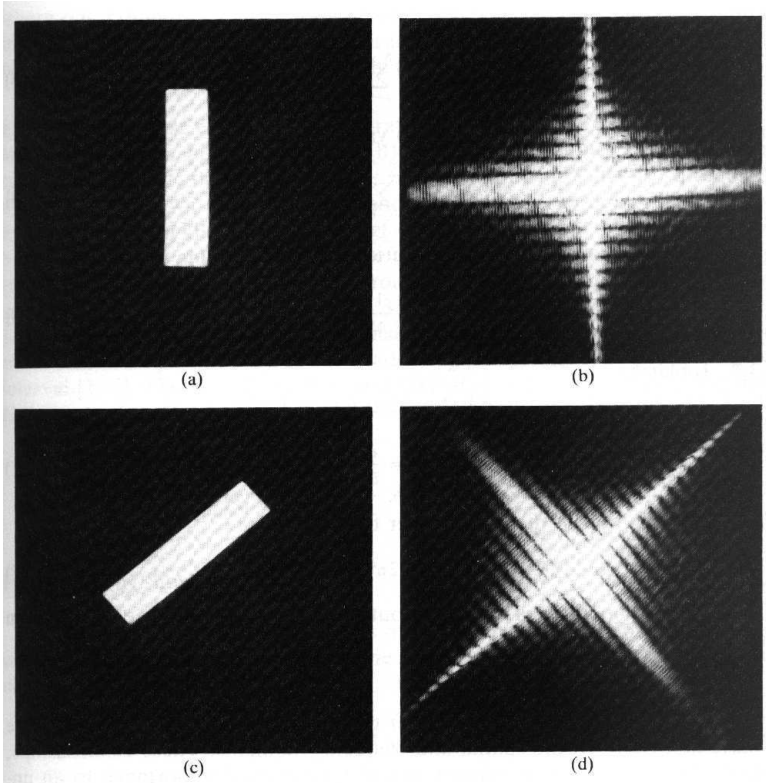
Properties of the Continuous Fourier Transform (3)



Rotation invariance of the Fourier transform. Authors: R. C. Gonzalez, R. E. Woods (1992)

Properties of the Continuous Fourier Transform (4)

- **Convolution Theorem**

  The convolution of two functions $f(x,y)$ and $g(x,y)$ is given by (cf. Lecture 2)

$$(f * g)(x,y) \;:=\; \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x-x', y-y')\, g(x',y')\, dx'\, dy'.$$

It forms the basis of all linear filters. For instance, for

$$g(x,y) \;:=\; \begin{cases} \frac{1}{\pi r^2} & \text{for } x^2+y^2 \le r^2, \\ 0 & \text{else,} \end{cases}$$

$f * g$ describes the smoothing of the image f by averaging all grey values within a neighbourhood of radius r. Computing this integral is expensive if r is large.

However, convolution is easily computed as multiplication in the Fourier domain:

$$\boxed{\mathcal{F}[f * g] \;=\; \mathcal{F}[f] \cdot \mathcal{F}[g]}$$

Afterwards the results must be transformed back to the spatial domain.

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- **Fourier Transform of a Gaussian**
  gives a Gaussian with reciprocal variance:

$$f(x,y) \;:=\; \exp\left(\frac{-\pi(x^2+y^2)}{\sigma^2}\right) \quad\Longrightarrow\quad \hat{f}(u,v) \;=\; \exp\left(\frac{-\pi(u^2+v^2)}{\sigma^{-2}}\right)$$

- **Fourier Transform of a Delta Comb**

  FT of an infinitely extended comb of delta pulses with peak distance λ,

$$f(x) = \sum_{k=-\infty}^{\infty} \delta(x-k\lambda),$$

  is a delta comb with reciprocal peak distance:

$$\hat{f}(u) = \sum_{k=-\infty}^{\infty} \delta(u-k/\lambda).$$

  Important when sampling continuous signals.

Towards the Discrete Setting: Sampling Theorem

Aliasing Effect:

- If a high-frequent signal is sampled too coarsely, low-frequent artifacts arise.
- This is called *aliasing*, for images sometimes also *Moiré effect*.
- can be observed quite often, e.g.
  - if the resolution of a scanner is too low
  - when using inappropriate programmes for downsampling (such as `xv`)
  - when some internet browsers automatically scale down large images

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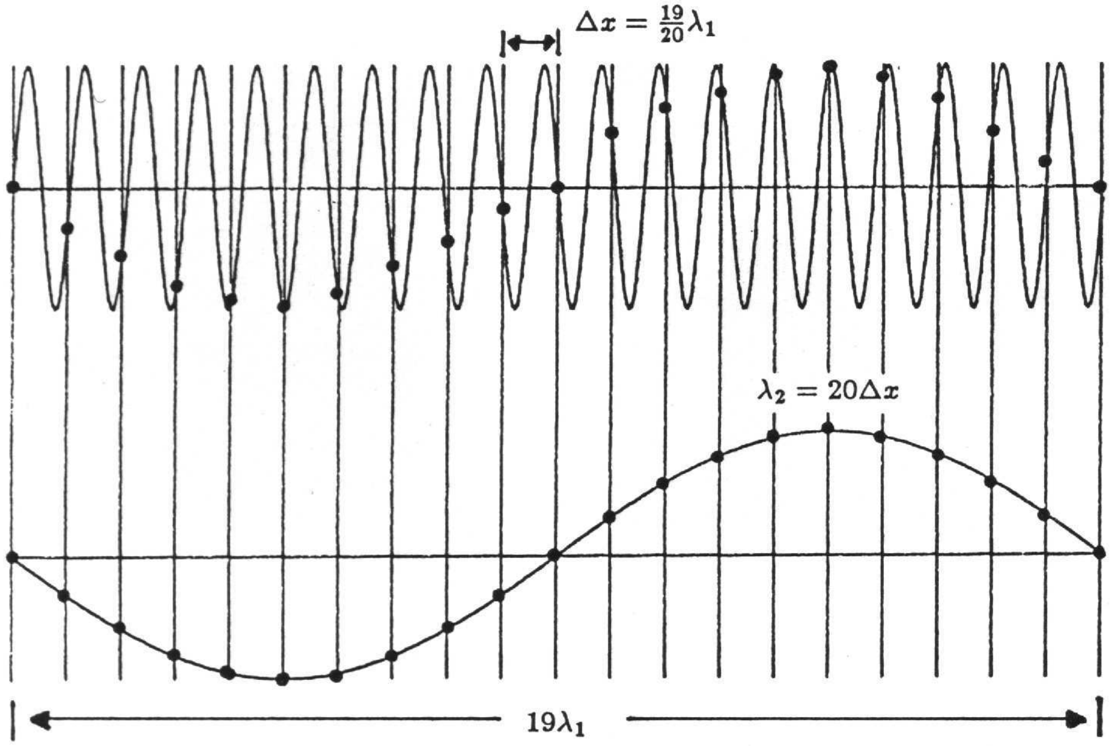
Towards the Discrete Setting: Sampling Theorem (2)



Aliasing effect. If the sampling rate is too low, artifacts arise. Author: B. Jähne (1991).

Towards the Discrete Setting: Sampling Theorem (3)

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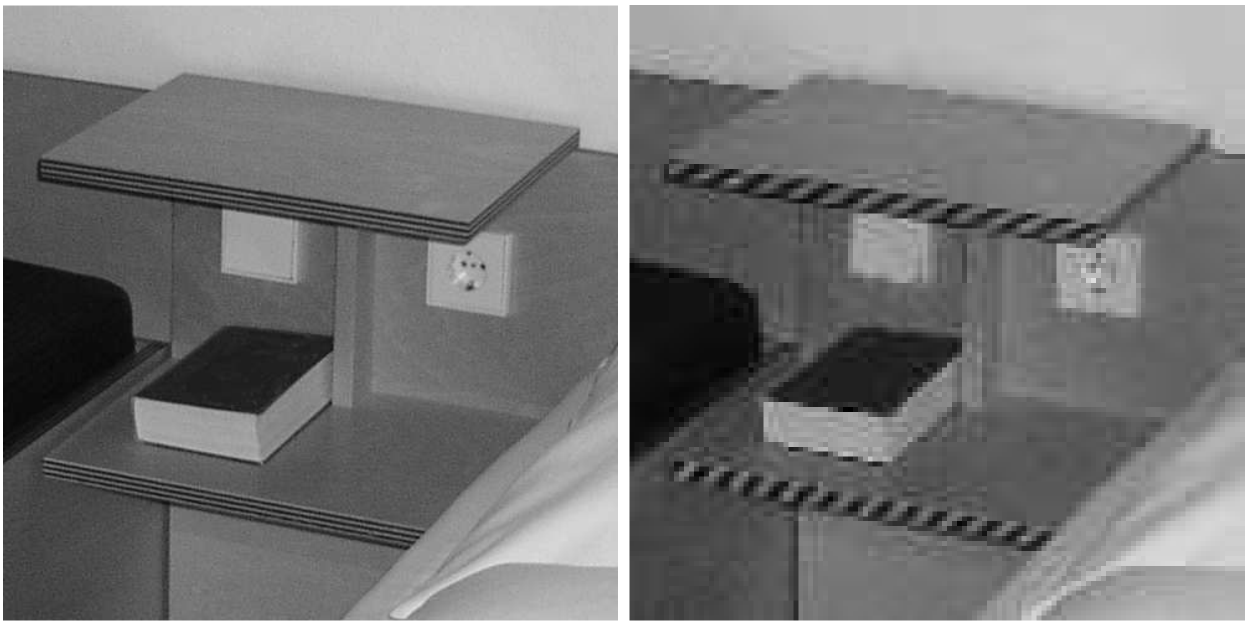

Aliasing effect. **(a) Left:** Original image, 496×496 pixels. **(b) Right:** Downsampled with xv to 124×124 pixels. Author: J. Weickert (2005).

Towards the Discrete Setting: Sampling Theorem (4)

Sampling Theorem (Abtasttheorem)

(Whittaker 1915, Nyquist 1928, Kotelnikov 1933, Shannon 1949)

- Let a signal f be *band-limited*, i.e. there exists a highest frequency W:

$$\hat{f}(u) = 0 \quad \text{for } |u| > W.$$

- In order to sample a band-limited signal correctly, one has to sample the highest frequency more than twice per period:

$$\Delta x < \Delta x_{limit} = \frac{1}{2W}.$$

Towards the Discrete Setting: Sampling Theorem (5)

Remarks

- The critical frequency where aliasing starts for a given sampling rate is called *Nyquist frequency*.
- If the sampling theorem is obeyed it is even possible to reconstruct the entire continuous signal from its discrete samples !
- For images, the sampling theorem holds in both directions.

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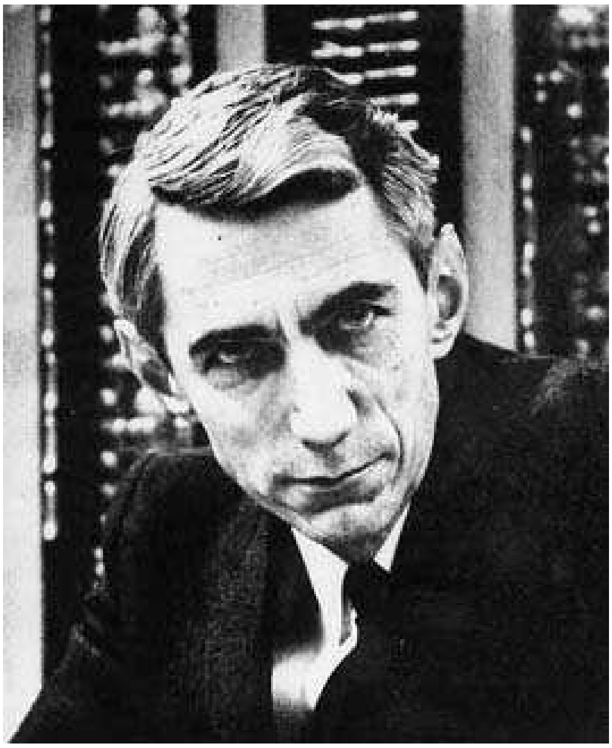
Towards the Discrete Setting: Sampling Theorem (6)



Claude E. Shannon (1916–2001) is the founder of information theory. He introduced the word "bit", devised the first chess-playing programmes, and was involved in the discovery of the sampling theorem. Source: `www-gap.dcs.st-and.ac.uk/~history/Mathematicians/Shannon.html`.

Towards the Discrete Setting: Sampling Theorem (7)

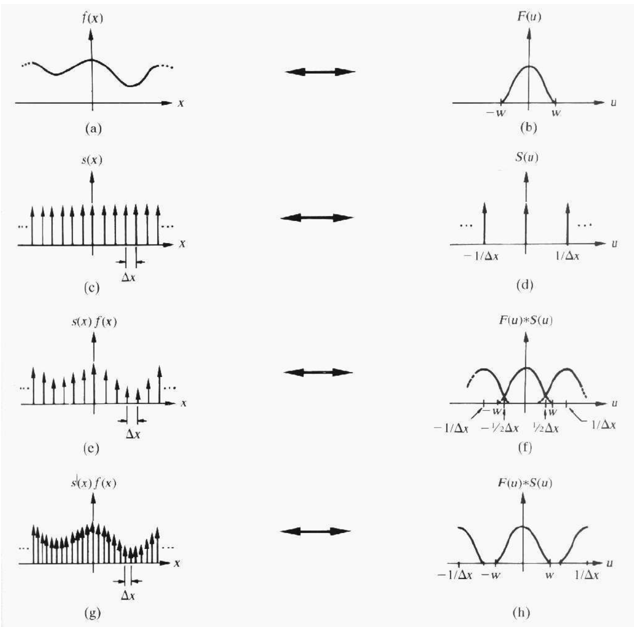


Illustration of the sampling theorem. (a) Band-limited function. (b) Fourier spectrum. (c) Delta comb. (d) FT of the delta comb is a delta comb with reciprocal grid distance. (e) Sampling a band-limited function is multiplication with a delta comb in the spatial domain. (f) In the Fourier domain this gives convolution of the Fourier transforms of (b) and (d). Overlapping frequency bands from different periods create aliasing. (g) Reduction of the sampling distance. (h) In the Fourier domain the frequency bands do no longer overlap and no aliasing effects arise. Authors: R. C. Gonzalez, R. E. Woods (1992).

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Towards the Discrete Setting: Sampling Theorem (8)

How Can One Avoid Aliasing ?

- Use a sufficiently high sampling rate.
- If this is not possible, suppress high frequencies by smoothing your image (e.g. by Gaussian convolution) *before* downsampling.

Towards the Discrete Setting: Sampling Theorem (9)

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Aliasing. **(a) Left:** Original image (496×496 pixels) after suppression of high frequencies by Gaussian convolution with $\sigma = 2$. **(b) Right:** Downsampling with xv to 124×124 pixels does not create visible aliasing effects in this case. Author: J. Weickert (2005).

Summary (1)

Summary

- The continuous Fourier transform analyses the frequency content of images.
- It is complex-valued, linear and separable.
- Spatial and Fourier domain are reciprocal with respect to localisation and orientation.
- Convolution in one domain becomes multiplication in the other.
- The Fourier transform maps box functions to sinc functions and Gaussians to Gaussians with reciprocal variance.
- A frequency must be sampled more than twice per period in order to avoid aliasing.

Summary (2)

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1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36

Literature

- R.C. Gonzalez, R.E. Woods: *Digital Image Processing*. Prentice Hall, Upper Saddle River, Second Edition, 2002.
  (a good textbook with in-depth introduction to the FT)
- T. Butz: *Fouriertransformation für Fußgänger*. Teubner, Stuttgart, 2005.
  (for those who wish to learn just a little bit more, but fear the full story)
- R. Bracewell: *The Fourier Transform and its Applications*. McGraw-Hill, New York, 1986.
  (the classical reference when you want to learn the full story about the FT)
- I. Amidror: *The Theory of the Moiré Phenomenon. Volume I: Periodic Layers*. Springer, Dordrecht, 2000.
  (an entire book on aliasing!)
- I. Amidror: *The Theory of the Moiré Phenomenon. Volume II: Aperiodic Layers*. Springer, Dordrecht, 2007.
  (further extensions on aliasing)
- Wikipedia: The Nyquist-Shannon Sampling Theorem.
  `http://en.wikipedia.org/wiki/Nyquist-Shannon_sampling_theorem`
  (contains proofs and useful links)

Assignment C1 – Classroom Work

Problem 1 (Box-Muller Algorithm)

Why do the numbers created by the Box-Muller algorithm obey a normal distribution ?

Since Gaussian noise is the most frequent noise in image processing, it is very important to understand how such noise can be created

Problem 2 (Convolution)

(a) Compute the convolution of a Heaviside function

$$H(x) := \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

with a truncated quadratic function

$$g(x) := \begin{cases} \frac{3}{4}(1-x^2) & (-1 \leq x \leq 1) \\ 0 & \text{(else)} \end{cases} \quad .$$

Sketch the two functions and their convolution result. This shows you what can happen to edges when they are convolved with a smoothing kernel.

(b) Assume $f \in C^0(\mathbb{R})$ is continuous and $g \in C^n(\mathbb{R})$ is n times continuously differentiable. Show that the convolution result $f * g$ is n times differentiable, too, i.e. $f * g \in C^n(\mathbb{R})$.
(*Hint:* Show first that the differentiation rule $(f * g)' = f' * g = f * g'$ holds.)

Since convolution is a frequent tool in signal and image processing, it is important to know its impacts, e.g. its regularising effect on the data.

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Assignment T1 – Theoretical Homework

Problem 1 (Signal-to-Noise Ratio) (4 points)

Let the image $f = (f_{i,j})$ be a noisy version of $g = (g_{i,j})$ degraded by additive noise $n = (n_{i,j})$ with zero mean:

$$f_{i,j} = g_{i,j} + n_{i,j} \ .$$

In the lecture the signal-to-noise ratio (SNR) is defined as a measure of quality.

(a) In which case do we have $\text{SNR}(f, g) = 0$? Please characterise this case mathematically.

(b) Let a filtered version u of f be given such that

$$\text{SNR}(u, g) = \text{SNR}(f, g) + 10\,\text{dB} \ .$$

How has the noise variance changed during the filtering?

(c) Consider two special cases in the calculations of $\text{SNR}(f, g)$:

- What happens if the initial image g is constant?
- In the case of a non-constant image g, try to calculate $\text{SNR}(g, g)$. How can you interpret this?

(*This exercise should give you deeper insights into the meaning of values obtained by the SNR.*

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Problem 2 (Discrete vs. Continuous Convolution) (6 points)

(a) Consider the discrete signal $f = (f_i)_{i \in \mathbb{Z}}$

$$f_i = \begin{cases} \frac{1}{2} & (i \in \{0, 1\}) \\ 0 & (\text{else}) \end{cases}$$

and convolve it three times with itself.

(b) Consider the continuous signal

$$f(x) = \begin{cases} \frac{1}{2} & (-1 \leq x \leq 1) \\ 0 & (\text{else}) \end{cases}$$

and convolve it three times with itself.

(This problem will be useful for approximating Gaussian convolution.)

Assignment T1 (3)

M I A 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36

Problem 3 (Properties of the Discrete Convolution) (5 points)

Show that the discrete convolution in 1-D,

$$(f * w)_i := \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k$$

of infinite discrete signals $f = (f_i)$, $g = (g_i)$ and $w = (w_i)$ possesses the following properties:

(a) Linearity: $(\alpha \cdot f + \beta \cdot g) * w = \alpha \cdot (f * w) + \beta \cdot (g * w) \quad$ for all $\alpha, \beta \in \mathbb{R}$.

(b) Commutativity: $f * w = w * f$.

(c) Identity: For which signal e does $f * e = f$ hold?

(*Since convolution is the central model for blur and will also be of fundamental importance in linear system theory, it is very useful to know these rules.*)

Assignment T1 (4)

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Problem 4 (Continuous Fourier Transform) (5 points)

Consider the following 1-D hat function:

$$f(x) = \begin{cases} \frac{2-|x|}{4} & (-2 \leq x \leq 2) \\ 0 & (\text{else}) \end{cases} .$$

(a) Compute the Fourier transform $\hat{f}(u) = \mathcal{F}[f]$ of f.

(b) Compute the corresponding Fourier spectrum $|\hat{f}(u)|$.

(*Computing the Fourier transform is one of the most important tasks in signal and image processing, since many filters are designed in the Fourier domain.*)

Deadline for submission: Tuesday, November 6, 10 am (before the lecture).