Image Processing and Computer Vision

Assignment T2

Jan Hendrik Dithmar
2031259

Group T1: Tue, 14-16 (Sebastian Zimmer)

This is a sample text to see, whether deactivating oplocks helps... blubb... blubb again

2.1 Image Pyramids

We have the following signal

$$f = (-20, -2, 11, -4, -12, -20, 6, 24, 12)^\top$$

Let's do the Gaussian pyramid first:

$$\begin{aligned}
v^3 &= f \\
v^2 &= R_3^2 v^3 \\
&= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -20 \\ -2 \\ 11 \\ -4 \\ -12 \\ -20 \\ 6 \\ 24 \\ 12 \end{pmatrix} \\
&= (-14, 4, -12, 4, 16)^\top \\
v^1 &= R_2^1 v^2 \\
&= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0\frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} \\
&= (-8, -4, 12)^\top \\
v^0 &= R_1^0 v^1 \\
&= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
&= 0
\end{aligned}$$

The Gaussian pyramid of the signal f is $\{v^3, v^2, v^1, v^0\}$.

Now I compute the Laplacian pyramid:

$$
\begin{aligned}
w^0 &= v^0 = 0 \\
w^1 &= v^1 - P_0^1 v^0 \\
&= \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot 0 \\
&= \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
w^2 &= v^2 - P_1^2 v^1 \\
&= \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
&= \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} - \begin{pmatrix} -8 \\ -6 \\ -4 \\ 4 \\ 12 \end{pmatrix} \\
&= \begin{pmatrix} -6 \\ 10 \\ -8 \\ 0 \\ 4 \end{pmatrix}
\end{aligned}
$$

$$\begin{aligned}
w^3 &= v^3 - P_2^3 v^2 \\
&= \begin{pmatrix} -20 \\ -2 \\ 1 \\ -4 \\ -12 \\ -20 \\ 6 \\ 24 \\ 12 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} \\
&= \begin{pmatrix} -20 \\ -2 \\ 1 \\ -4 \\ -12 \\ -20 \\ 6 \\ 24 \\ 12 \end{pmatrix} - \begin{pmatrix} -14 \\ -5 \\ 4 \\ -4 \\ -12 \\ -4 \\ 4 \\ 10 \\ 16 \end{pmatrix} \\
&= (-6, 3, 7, 0, 0, -16, 2, 14, -4)^\top
\end{aligned}$$

The Laplacian pyramid of the signal f is $\{w^3, w^2, w^1, w^0\}$.

Now I reconstruct the initial signal from the Laplacian pyramid:

$$\begin{aligned}
v^0 &= w^0 = 0 \\
v^1 &= w^1 + P_0^1 v^0 \\
&= \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
v^2 &= w^2 + P_1^2 v^1 \\
&= \begin{pmatrix} -6 \\ 10 \\ -8 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
&= \begin{pmatrix} -6 \\ 10 \\ -8 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -8 \\ -6 \\ -4 \\ 4 \\ 12 \end{pmatrix} \\
&= (-14, 4, -12, 4, 16)^\top
\end{aligned}
$$

$$
\begin{aligned}
v^3 &= w^3 + P_2^3 v^2 \\
&= \begin{pmatrix} -6 \\ 3 \\ 7 \\ 0 \\ 0 \\ -16 \\ 2 \\ 14 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} \\
&= \begin{pmatrix} -6 \\ 3 \\ 7 \\ 0 \\ 0 \\ -16 \\ 2 \\ 14 \\ -4 \end{pmatrix} + \begin{pmatrix} -14 \\ -5 \\ 4 \\ -4 \\ -12 \\ -4 \\ 4 \\ 10 \\ 16 \end{pmatrix} \\
&= (-20, -2, 11, -4, -12, -20, 6, 24, 12)^\top = f
\end{aligned}
$$

2.2 Discrete Fourier Transform

First of all, I want to rename the signal f_1 to f and f_2 to g for the sake of readability when it comes to (double-)indicies later on. So we have:

$$f = (6, 8, 2, 4)^\top \quad \text{and} \quad g = (4, 6, 8, 2)^\top$$

a.

$$\begin{aligned}
\hat{f}_0 &= \frac{1}{2} \cdot \sum_{m=0}^{3} f_m \cdot \exp\left(-\frac{i2\pi 0m}{4}\right) \\
&= \frac{1}{2} \cdot (f_0 + f_1 + f_2 + f_3) \\
&= \frac{1}{2} \cdot (6 + 8 + 2 + 4) \\
&= 10
\end{aligned}$$

$$\begin{aligned}
\hat{f}_1 &= \frac{1}{2} \cdot \sum_{m=0}^{3} f_m \cdot \exp\left(-\frac{i2\pi 1m}{4}\right) \\
&= \frac{1}{2} \cdot (f_0 \cdot e^0 + f_1 \cdot e^{-i\pi/2} + f_2 \cdot e^{-i\pi} + f_3 \cdot e^{-i3\pi/2}) \\
&= \frac{1}{2} \cdot (6 \cdot 1 + 8 \cdot (-i) + 2 \cdot (-1) + 4 \cdot i) = \frac{4 - 4i}{2} \\
&= 2 - 2i
\end{aligned}$$

$$\begin{aligned}
\hat{f}_2 &= \frac{1}{2} \cdot \sum_{m=0}^{3} f_m \cdot \exp\left(-\frac{i2\pi 2m}{4}\right) \\
&= \frac{1}{2} \cdot (f_0 \cdot e^0 + f_1 \cdot e^{-i\pi} + f_2 \cdot e^{-i2\pi} + f_3 \cdot e^{-i3\pi}) \\
&= \frac{1}{2} \cdot (6 \cdot 1 + 8 \cdot (-1) + 2 \cdot 1 + 4 \cdot (-1)) \\
&= -2
\end{aligned}$$

$$\begin{aligned}
\hat{f}_3 &= \frac{1}{2} \cdot \sum_{m=0}^{3} f_m \cdot \exp\left(-\frac{i2\pi 3m}{4}\right) \\
&= \frac{1}{2} \cdot (f_0 \cdot e^0 + f_1 \cdot e^{-i3\pi/2} + f_2 \cdot e^{-i3\pi} + f_3 \cdot e^{-i9\pi/2}) \\
&= \frac{1}{2} \cdot (6 \cdot 1 + 8 \cdot i + 2 \cdot (-1) + 4 \cdot (-i)) = \frac{4 + 4i}{2} \\
&= 2 + 2i
\end{aligned}$$

$$\begin{aligned}
\hat{g}_0 &= \frac{1}{2} \cdot \sum_{m=0}^{3} g_m \cdot \exp\left(-\frac{i2\pi 0m}{4}\right) \\
&= \frac{1}{2} \cdot (g_0 + g_1 + g_2 + g_3) = \frac{1}{2} \cdot (4 + 6 + 8 + 2) \\
&= 10
\end{aligned}$$

$$\begin{aligned}
\hat{g}_1 &= \frac{1}{2} \cdot \sum_{m=0}^{3} g_m \cdot \exp\left(-\frac{i2\pi 1m}{4}\right) \\
&= \frac{1}{2} \cdot (g_0 \cdot e^0 + g_1 \cdot e^{-i\pi/2} + g_2 \cdot e^{-i\pi} + g_3 \cdot e^{-i3\pi/2}) \\
&= \frac{1}{2} \cdot (4 \cdot 1 + 6 \cdot (-i) + 8 \cdot (-1) + 2 \cdot i) = \frac{-4 - 4i}{2} \\
&= -2 - 2i
\end{aligned}$$

$$\begin{aligned}
\hat{g}_2 &= \frac{1}{2} \cdot \sum_{m=0}^{3} g_m \cdot \exp\left(-\frac{i2\pi 2m}{4}\right) \\
&= \frac{1}{2} \cdot (g_0 \cdot e^0 + g_1 \cdot e^{-i\pi} + g_2 \cdot e^{-i2\pi} + g_3 \cdot e^{-i3\pi}) \\
&= \frac{1}{2} \cdot (4 \cdot 1 + 6 \cdot (-1) + 8 \cdot 1 + 2 \cdot (-1)) \\
&= 2
\end{aligned}$$

$$\begin{aligned}
\hat{g}_3 &= \frac{1}{2} \cdot \sum_{m=0}^{3} g_m \cdot \exp\left(-\frac{i2\pi 3m}{4}\right) \\
&= \frac{1}{2} \cdot (g_0 \cdot e^0 + g_1 \cdot e^{-i3\pi/2} + g_2 \cdot e^{-i3\pi} + g_3 \cdot e^{-i9\pi/2}) \\
&= \frac{1}{2} \cdot (4 \cdot 1 + 6 \cdot i + 8 \cdot (-1) + 2 \cdot (-i)) = \frac{-4 + 4i}{2} \\
&= -2 + 2i
\end{aligned}$$

Now I compute the spectra of the signals:

$$\begin{aligned}
\left|\hat{f}_0\right| &= \sqrt{10^2} = 10 \\
\left|\hat{f}_1\right| &= \sqrt{2^2 + (-2 \cdot i)^2} = \sqrt{4 + 4i^2} = \sqrt{4 - 4} = 0 \\
\left|\hat{f}_2\right| &= \sqrt{(-2)^2} = 2 \\
\left|\hat{f}_3\right| &= \sqrt{2^2 + (2 \cdot i)^2} = \sqrt{4 + 4i^2} = \sqrt{4 - 4} = 0
\end{aligned}$$

$$\begin{aligned}
|\hat{g}_0| &= \sqrt{10^2} = 10 \\
|\hat{g}_1| &= \sqrt{(-2)^2 + (-2 \cdot i)^2} = \sqrt{4 + 4i^2} = \sqrt{4 - 4} = 0 \\
|\hat{g}_2| &= \sqrt{2^2} = 2 \\
|\hat{g}_3| &= \sqrt{(-2)^2 + (2 \cdot i)^2} = \sqrt{4 + 4i^2} = \sqrt{4 - 4} = 0
\end{aligned}$$

The Fourier spectra of f and g are identical while the coefficients of f and g are different.

b. The highest frequency is 10 which I set to 0, so I obtain:

$$\begin{aligned}
\hat{f}_0 &= 0 \\
\hat{f}_1 &= 2 - 2i \\
\hat{f}_2 &= -2 \\
\hat{f}_3 &= 2 + 2i
\end{aligned}$$

The backtransform looks like this:

$$\begin{aligned}
f_0 &= \frac{1}{2} \cdot \sum_{p=0}^{3} \hat{f}_p \cdot \exp\left(\frac{i2\pi p0}{4}\right) \\
&= \frac{1}{2} \cdot (0 + (2 - 2i) - 2 + (2 + 2i)) \\
&= \frac{1}{2} \cdot 2 \\
&= 1
\end{aligned}$$

$$
\begin{aligned}
f_1 &= \frac{1}{2} \cdot \sum_{p=0}^{3} \hat{f}_p \cdot \exp\left(\frac{i2\pi p1}{4}\right) \\
&= \frac{1}{2} \cdot (0 + (2-2i) \cdot e^{i\pi/2} - 2 \cdot e^{i\pi} + (2+2i) \cdot e^{i3\pi/2}) \\
&= \frac{1}{2} \cdot ((2-2i) \cdot (-i) - 2 \cdot (-1) + (2+2i) \cdot i) \\
&= \frac{1}{2} \cdot (-2i + 2i^2 + 2 + 2i + 2i^2) \\
&= \frac{1}{2} \cdot (-2) \\
&= -1
\end{aligned}
$$

$$
\begin{aligned}
f_2 &= \frac{1}{2} \cdot \sum_{p=0}^{3} \hat{f}_p \cdot \exp\left(\frac{i2\pi p2}{4}\right) \\
&= \frac{1}{2} \cdot (0 + (2-2i) \cdot e^{i\pi} - 2 \cdot e^{i2\pi} + (2+2i) \cdot e^{i3\pi}) \\
&= \frac{1}{2} \cdot ((2-2i) \cdot (-1) - 2 + (2+2i) \cdot (-1)) \\
&= \frac{1}{2} \cdot (-2 + 2i - 2 - 2 - 2i) \\
&= \frac{1}{2} \cdot (-6) \\
&= -3
\end{aligned}
$$

$$
\begin{aligned}
f_3 &= \frac{1}{2} \cdot \sum_{p=0}^{3} \hat{f}_p \cdot \exp\left(\frac{i2\pi p3}{4}\right) \\
&= \frac{1}{2} \cdot (0 + (2-2i) \cdot e^{i3\pi/2} - 2 \cdot e^{i3\pi} + (2+2i) \cdot e^{i9\pi/2}) \\
&= \frac{1}{2} \cdot ((2-2i) \cdot i + 2 + (2+2i) \cdot (-i)) \\
&= \frac{1}{2} \cdot (2i - 2i^2 + 2 - 2i - 2i^2) \\
&= \frac{1}{2} \cdot 6 \\
&= 3
\end{aligned}
$$

The resulting signal is $f = (1, -1, -3, 3)^\top$. This signal is symmetric which results in a better signal with respect to smoothness.

2.3 Discrete Fourier Transform

To show that

$$v_p = \frac{1}{\sqrt{M}} \left(\exp\left(\frac{2\pi i p 0}{M}\right), \exp\left(\frac{2\pi i p 1}{M}\right), \ldots, \exp\left(\frac{2\pi i p (M-1)}{M}\right) \right)^\top$$

form an orthonormal basis one has to show the following:

1. We have M different vectors (*already known*)
2. $v_i \in \mathbb{C}^M \; \forall i \in \{0, ..., M-1\}$ (*already known*)
3. Vectors are orthogonal and have norm 1 which meands that they are orthonormal.

The last point has to be shown. For this proof I want to state the following:

$$\langle v_p, v_q \rangle = \begin{cases} 1 & (p = q) \\ 0 & (\text{else}) \end{cases} \qquad \forall p, q \in \{0, ..., M-1\}$$

To use this fact, a case distinction is necessary:

- Let $p \in \{0, ..., M-1\}, p = q$.
  Let's plug in the definition:

$$\begin{aligned}
\langle v_p, v_q \rangle &= \sum_{m=0}^{M-1} \frac{1}{\sqrt{M}} \cdot \exp\left(\frac{i2\pi pm}{M}\right) \cdot \overline{\frac{1}{\sqrt{M}} \cdot \exp\left(\frac{i2\pi qm}{M}\right)} \\
&= \sum_{m=0}^{M-1} \frac{1}{\sqrt{M}} \cdot \exp\left(\frac{i2\pi pm}{M}\right) \cdot \frac{1}{\sqrt{M}} \cdot \exp\left(-\frac{i2\pi qm}{M}\right) \\
&= \frac{1}{M} \cdot \sum_{m=0}^{M-1} \cdot \exp\left(\frac{i2\pi m}{M} \cdot (p - q)\right) \\
&\overset{(p=q)}{=} \frac{1}{M} \cdot \sum_{m=0}^{M-1} \underbrace{\exp(0)}_{=1} \\
&= \frac{M}{M} \\
&= 1
\end{aligned}$$

- Let $p \in \{0, ..., M-1\}, p \neq q$.
  Let's plug in the definition again:

$$\begin{aligned}
\langle v_p, v_q \rangle &= \sum_{m=0}^{M-1} \frac{1}{\sqrt{M}} \cdot \exp\left(\frac{i2\pi pm}{M}\right) \cdot \overline{\frac{1}{\sqrt{M}} \cdot \exp\left(\frac{i2\pi qm}{M}\right)} \\
&= \sum_{m=0}^{M-1} \frac{1}{\sqrt{M}} \cdot \exp\left(\frac{i2\pi pm}{M}\right) \cdot \frac{1}{\sqrt{M}} \cdot \exp\left(-\frac{i2\pi qm}{M}\right) \\
&= \frac{1}{M} \cdot \sum_{m=0}^{M-1} \cdot \exp\left(\frac{i2\pi m}{M} \cdot (p-q)\right) \\
&= \frac{1}{M} \cdot \sum_{m=0}^{M-1} \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)^m \\
&\overset{(*)}{=} \frac{1}{M} \cdot \frac{1 - \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)^M}{1 - \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)} \\
&= \frac{1}{M} \cdot \frac{1 - \exp\left(\frac{i2\pi M}{M} \cdot (p-q)\right)}{1 - \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)} \\
&= \frac{1}{M} \cdot \frac{1 - \exp(i2\pi(p-q))}{1 - \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)} \\
&= \frac{1}{M} \cdot \frac{1 - \exp(i2\pi)^{(p-q)}}{1 - \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)} \\
&\overset{(**)}{=} \frac{1}{M} \cdot \frac{1 - 1^{(p-q)}}{1 - \exp\left(\frac{i2\pi}{M} \cdot (p-q)\right)} \\
&= 0
\end{aligned}$$

  where $(*)$ is the formula for the geometric series

$$\sum_{k=0}^{n} r^k = \frac{1 - r^{n+1}}{1 - r} \quad (r \neq 1)$$

  and $(**)$ is the fact that $1 = e^{i2\pi} \ \forall n \in \mathbb{Z}$. I want to prove the latter in just a few lines:

$$\begin{aligned}
1 &= 1^n \\
&= \left(e^{i2\pi}\right)^n \\
&= e^{i2\pi n}
\end{aligned}$$

  Since $(p-q) \in \mathbb{Z}$, the upper equations hold for our case.