Image Processing and Computer Vision

Assignment T5

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Group T1: Tue, 14-16 (Sebastian Zimmer)

5.1 Derivative Filters

a. The equation system looks as follows:

$$\begin{aligned}
f_{i-2} &= f_i - 2hf_i' + 4\frac{h^2}{2}f_i'' - 8\frac{h^3}{6}f_i''' + 16\frac{h^4}{24}f_i'''' + O(h^5) \\
&= f_i - 2hf_i + 2h^2f_i' - \frac{4}{3}h^3f_i''' + \frac{2}{3}h^4f_i'''' + O(h^5) \\
f_{i-1} &= f_i - hf_i' + \frac{1}{2}h^2f_i'' - \frac{1}{6}h^3f_i''' + \frac{1}{24}h^4f_i'''' + O(h^5) \\
f_i &= f_i \\
f_{i+1} &= f_i + hf_i' + \frac{1}{2}h^2f_i'' + \frac{1}{6}h^3f_i''' + \frac{1}{24}h^4f_i'''' + O(h^5) \\
f_{i+2} &= f_i + 2hf_i' + 4\frac{h^2}{2}f_i'' + 8\frac{h^3}{6}f_i''' + 16\frac{h^4}{24}f_i'''' + O(h^5) \\
&= f_i + 2hf_i + 2h^2f_i' + \frac{4}{3}h^3f_i''' + \frac{2}{3}h^4f_i'''' + O(h^5)
\end{aligned}$$

$$\Rightarrow 0 \cdot f_i + 1 \cdot f_i' + 0 \cdot f_i'' = \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2}$$

If you rewrite this, you get the following:

$$\begin{aligned}
f_i' &= \alpha_{-2} \cdot \left(f_i - 2hf_i' + 2h^2f_i'' - \frac{4}{3}h^3f_i''' + \frac{2}{3}h^4f_i'''' + O(h^5)\right) \\
&\quad + \alpha_{-1} \cdot \left(f_i - hf_i' + \frac{1}{2}h^2f_i'' - \frac{1}{6}h^3f_i''' + \frac{1}{24}h^4f_i'''' + O(h^5)\right) \\
&\quad + \alpha_0 \cdot f_i \\
&\quad + \alpha_1 \cdot \left(f_i + hf_i' + \frac{1}{2}h^2f_i'' + \frac{1}{6}h^3f_i''' + \frac{1}{24}h^4f_i'''' + O(h^5)\right) \\
&\quad + \alpha_2 \cdot \left(f_i + 2hf_i' + 2h^2f_i'' + \frac{4}{3}h^3f_i''' + \frac{2}{3}h^4f_i'''' + O(h^5)\right) \\
&= (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2)f_i + (-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2)hf_i' \\
&\quad + \left(2\alpha_{-2} + \frac{1}{2}\alpha_{-1} + \frac{1}{2}\alpha_1 + 2\alpha_2\right)h^2f_i'' \\
&\quad + \left(-\frac{4}{3}\alpha_{-2} - \frac{1}{6}\alpha_{-1} + \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2\right)h^3f_i''' \\
&\quad + \left(\frac{2}{3}\alpha_{-2} + \frac{1}{24}\alpha_{-1} + \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2\right)h^4f_i'''' + O(h^5)
\end{aligned}$$

This leads to the linear system of equations:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

b. As in the script, I extend the Taylor expansion by one degree:

$$\begin{aligned}
f_{i-2} &= f_i - 2hf_i' + 2h^2 f_i'' - \frac{4}{3}h^3 f_i''' + \frac{2}{3}h^4 f_i'''' - \frac{4}{15}h^5 f_i''''' + O(h^6) \\
f_{i-1} &= f_i - hf_i' + \frac{1}{2}h^2 f_i'' - \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i'''' - \frac{1}{120}h^5 f_i''''' + O(h^6) \\
f_i &= f_i \\
f_{i+1} &= f_i + hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i'''' + \frac{1}{120}h^5 f_i''''' + O(h^6) \\
f_{i+2} &= f_i + 2hf_i' + 2h^2 f_i'' + \frac{4}{3}h^3 f_i''' + \frac{2}{3}h^4 f_i'''' + \frac{4}{15}h^5 f_i''''' + O(h^6)
\end{aligned}$$

Using the filter mask, one gets:

$$\begin{aligned}
f_i' &= \frac{1}{12h}\Bigg[(1-8+8-1)f_i + (-2+8+8-2)hf_i' \\
&\qquad + (2-4+4-2)h^2 f_i'' \\
&\qquad + \left(-\frac{4}{3}+\frac{4}{3}+\frac{4}{3}-\frac{4}{3}\right)h^3 f_i''' \\
&\qquad + \left(\frac{2}{3}-\frac{1}{3}+\frac{1}{3}-\frac{2}{3}\right)h^4 f_i'''' \\
&\qquad + \left(-\frac{4}{15}+\frac{1}{15}+\frac{1}{15}-\frac{4}{15}\right)h^5 f_i''''' \\
&\qquad + O(h^6)\Bigg] \\
&= \frac{1}{12h}\left[12hf_i' - \frac{2}{5}h^5 f_i''''' + O(h^6)\right] \\
f_i' &= f_i' - \frac{1}{30}h^4 f_i''''' + O(h^5)
\end{aligned}$$

The order of consistency for this approximation is 4.

c. If I want to approximate a derivative of order d with n points, it must be that $n > d$.

For a value j, where $j \neq d$ and $j < n$, all the coefficients in front of $h^j f_i^{(j)}$ are 0 (this is also the case in part **b.**). What remains is $h^d f_i^{(d)}$ and $h^n f_i^{(n)}$ with their respective coefficients.
The order of consistency is therefore $n - d$. If you compare this statement with my solution of part **b.**, you see that it is true, since we have order of consistency $n - d = 5 - 1 = 4$.

5.2 Hough Transform

a. First of all, I have to state the five trigonometric curves which belong to the five points given.

1. $(-1, 1)$: $d = -\cos\phi + \sin\phi$
2. $(2, 0)$: $d = 2 \cdot \cos\phi$
3. $(3, 1)$: $d = 3 \cdot \cos\phi + \sin\phi$
4. $(1, 3)$: $d = \cos\phi + 3 \cdot \cos\phi$
5. $(0, 2)$: $d = 2 \cdot \sin\phi$

Now I try to find the points which are on a line by using equality:

1. = 2.:

$$\begin{aligned} -\cos\phi + \sin\phi &= 2 \cdot \cos\phi \\ \sin\phi &= 3 \cdot \cos\phi \end{aligned}$$

1. = 3.:

$$\begin{aligned} -\cos\phi + \sin\phi &= 3 \cdot \cos\phi + \sin\phi \\ 4 \cdot \cos\phi &= 0 \end{aligned}$$

1. = 4.:

$$\begin{aligned} -\cos\phi + \sin\phi &= \cos\phi + 3 \cdot \sin\phi \\ -\cos\phi &= \sin\phi \end{aligned}$$

1. = 5.:

$$\begin{aligned} -\cos\phi + \sin\phi &= 2 \cdot \sin\phi \\ -\cos\phi &= \sin\phi \end{aligned}$$

It follows that the points $(-1, 1)$, $(1, 3)$ and $(0, 2)$ belong to the same line.

b. The equation for a circle looks like this:

$$|x - a|^2 + |y - b|^2 - r^2 = 0$$

where (a, b) is the center and r the radius of the circle. For this exercise, we have $r = 2 \Rightarrow r^2 = 4$.
I will now look at the different points:

1. $(-1, 1)$: $|x + 1|^2 + |y - 1|^2 = 4$
2. $(2, 0)$: $|x - 2|^2 + |y - 0|^2 = 4$
3. $(3, 1)$: $|x - 3|^2 + |y - 1|^2 = 4$
4. $(1, 3)$: $|x - 1|^2 + |y - 3|^2 = 4$
5. $(0, 2)$: $|x - 0|^2 + |y - 2|^2 = 4$

As one can easily see, the points 1, 3 and 4 belong to the same circle with radius 2 and the center point $(1, 1)$.

5.3 Cooccurrence Matrices

Since $d = (-1, -1)^\top$, we have to look at the upper left neighbor.

$$\frac{1}{30} \times \begin{array}{c|c} \multicolumn{2}{c}{\text{P[i,j]}} \\ \begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 5 & 1 & 1 \\ 2 & 2 & 3 & 2 \\ 1 & 2 & 2 & 2 \end{bmatrix} & \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} \quad i \\ \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} & \\ j & \end{array}$$