Image Processing and Computer Vision

Assignment T4

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Group T1: Tue, 14-16 (Sebastian Zimmer)

4.1 Continuous Nonquadratic Variational Methods

a. To state the Euler-Lagrange equations I have to compute the partial derivatives first, where I only look at the integrand

$$F(x,u,u_x) = \left(\frac{1}{2}(u-f)^2 + \alpha \cdot \lambda^2 \cdot \sqrt{1+\frac{u_x^2}{\lambda^2}}\right) dx$$

$$\begin{aligned}
F_u &= u - f \\
F_{u_x} &= \alpha \cdot \lambda^2 \cdot \frac{1}{2} \frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot \frac{2}{\lambda^2} \cdot u_x \\
&= \alpha \cdot \frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot u_x
\end{aligned}$$

Now I can state the Euler-Lagrange equation:

$$\begin{aligned}
0 &= F_u - \frac{d}{dx} F_{u_x} = (u-f) - \frac{d}{dx} \frac{\alpha}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot u_x \\
&= (u-f) - \alpha \frac{d}{dx} \frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot u_x \\
&= u - f - \alpha \cdot \left(\frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot u_{xx} + \frac{d}{dx} \cdot \left(\frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \right) \cdot u_x \right) \\
&= u - f - \alpha \cdot \left(\frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot u_{xx} + u_x \cdot \left(-\frac{1}{2}\right) \cdot 2 \cdot \frac{u_x \cdot u_{xx}}{\lambda^2 \cdot (1+\frac{u_x^2}{\lambda^2})^{\frac{3}{2}}} \right) \\
&= u - f - \alpha \cdot \left(\frac{1}{\sqrt{1+\frac{u_x^2}{\lambda^2}}} \cdot u_{xx} - u_x^2 \cdot u_{xx} \cdot \frac{1}{\lambda^2 \cdot (1+\frac{u_x^2}{\lambda^2})^{\frac{3}{2}}} \right)
\end{aligned}$$

$$
\begin{aligned}
&= u - f - \alpha \cdot u_{xx} \cdot \left(\frac{1}{\sqrt{1 + \frac{u_x^2}{\lambda^2}}} - u_x^2 \cdot \frac{1}{\lambda^2 \cdot (1 + \frac{u_x^2}{\lambda^2})^{\frac{3}{2}}} \right) \\
&= u - f - \alpha \cdot u_{xx} \cdot \left(\frac{\lambda^2 \cdot (1 + \frac{u_x^2}{\lambda^2})}{\lambda^2 \cdot (1 + \frac{u_x^2}{\lambda^2})^{\frac{3}{2}}} - u_x^2 \cdot \frac{1}{\lambda^2 \cdot (1 + \frac{u_x^2}{\lambda^2})^{\frac{3}{2}}} \right) \\
&= u - f - \alpha \cdot u_{xx} \cdot \left(\frac{\lambda^2 + u_x^2 - u_x^2}{\lambda^2 \cdot (1 + \frac{u_x^2}{\lambda^2})^{\frac{3}{2}}} \right) \\
&= u - f - \alpha \cdot u_{xx} \cdot \frac{1}{(1 + \frac{u_x^2}{\lambda^2})^{\frac{3}{2}}}
\end{aligned}
$$

with boundary conditions

$$F_{u_x} = 0$$

in $x = a$ and $x = b$.

b. λ is some kind of contrast parameter. The idea is that if $|\nabla u|$ is greater than a certain "threshold" λ, you have locations where you have edges. This means that you can use this fact for edge preserving.

c. First, one has to show that $E(u)$ is strictly convex. From the lecture (Lecture 15, slide 7) we know that it is helpful to use the fact that $g(s) = s^2$ is strictly convex. Furthermore, one has to show that

$$E(\beta u + (1-\beta)v) < \beta E(u) + (1-\beta)E(v)$$

which I will do first:

$$
\begin{aligned}
E(\beta u + (1-\beta)v) &= \int_a^b \left(\frac{1}{2}((\beta u + (1-\beta)v) - f)^2 \right. \\
&\qquad \left. + \alpha\lambda^2 \sqrt{1 + \frac{(\beta u_x + (1-\beta)v_x)^2}{\lambda^2}} \right) dx \\
&= \frac{1}{2} \cdot \int_a^b (\beta u + (1-\beta)v - f)^2 dx \\
&\qquad + \alpha\lambda^2 \int_a^b \sqrt{1 + \frac{(\beta u_x + (1-\beta)v_x)^2}{\lambda^2}} dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\int_a^b (\beta(u-f)+(1-\beta)(v-f))^2 dx \\
&\quad +\alpha\lambda^2\int_a^b \sqrt{1+\frac{(\beta u_x+(1-\beta)v_x)^2}{\lambda^2}}dx \\
&\overset{(*)}{<} \frac{1}{2}\beta\int_a^b (u-f)^2 dx \\
&\quad +\frac{1}{2}(1-\beta)\int_a^b (v-f)^2 dx \\
&\qquad +\alpha\lambda^2\int_a^b \sqrt{1+\frac{(\beta u_x+(1-\beta)v_x)^2}{\lambda^2}}dx \\
&\overset{(**)}{<} \frac{1}{2}\beta\int_a^b (u-f)^2 dx+\frac{1}{2}(1-\beta)\int_a^b (v-f)^2 dx \\
&\quad +\alpha\lambda^2\beta\int_a^b \sqrt{1+\frac{u_x^2}{\lambda^2}}dx \\
&\qquad +\alpha\lambda^2(1-\beta)\int_a^b \sqrt{1+\frac{v_x^2}{\lambda^2}}dx \\
&= \beta\int_a^b \left(\frac{1}{2}(u-f)^2+\alpha\lambda^2\sqrt{1+\frac{u_x^2}{\lambda^2}}\right)dx \\
&\quad +(1-\beta)\int_a^b \left(\frac{1}{2}(v-f)^2+\alpha\lambda^2\sqrt{1+\frac{v_x^2}{\lambda^2}}\right)dx \\
&= \beta\cdot E(u)+(1-\beta)\cdot E(v)
\end{aligned}
$$

where

- $(*)$: $g(s)=s^2$ is strictly convex.
- $(**)$: $g(y)=\sqrt{1+y^2}$ is strictly convex.

Proof:

$$\begin{aligned}
f(x) &= \sqrt{1+\frac{x^2}{\lambda^2}} = \sqrt{1+\left(\frac{x}{\lambda}\right)^2} \\
\text{Substitution: } y &= \frac{x}{\lambda} \\
\Rightarrow g(y) &= \sqrt{1+y^2} \\
\\
g_y &= \frac{y}{\sqrt{1+y^2}} \\
g_{yy} &= \frac{1}{(1+y^2)^{\frac{3}{2}}}
\end{aligned}$$

The second derivative is positive everywhere, so $g(y) = \sqrt{1+y^2}$ is strictly convex.

Result: This energy functional is strictly convex. From the lecture we know that the solution is unique and the single extremum of $E(u)$ is then a minimum.

4.2 Discrete Variational Methods

a.

$$E_f(u) = \frac{1}{2} \cdot \sum_{k=1}^{N} (u_k - f_k)^2 + \alpha \cdot \lambda^2 \cdot \sum_{k=1}^{N-1} \sqrt{1 + \frac{(u_{k+1}-u_k)^2}{\lambda^2 \cdot h^2}}$$

where the signal is given by $[u_1, \ldots, u_N]$ and h is the step-size.

b. We know from the lecture (Lecture 15, slide 5) that the first partial derivatives w.r.t. $u_1, ..., u_N$ must vanish:

$$\begin{aligned}
0 &= \frac{\partial E_f}{\partial u_1} = u_1 - f_1 + \frac{\alpha}{h^2} \cdot \frac{u_2 - u_1}{\sqrt{1 + \frac{(u_2-u_1)^2}{h^2 \cdot \lambda^2}}} \\
0 &= \frac{\partial E_f}{\partial u_i} = u_i - f_i + \frac{\alpha}{h^2} \cdot \left(\frac{u_i - u_{i-1}}{\sqrt{1 + \frac{(u_i-u_{i-1})^2}{h^2 \cdot \lambda^2}}} - \frac{u_{i+1} - u_i}{\sqrt{1 + \frac{(u_{i+1}-u_i)^2}{h^2 \cdot \lambda^2}}} \right) \\
& \qquad , i = 2, ..., N-1 \\
0 &= \frac{\partial E_f}{\partial u_N} = u_N - f_N + \frac{\alpha}{h^2} \cdot \frac{u_N - u_{N-1}}{\sqrt{1 + \frac{(u_N-u_{N-1})^2}{h^2 \cdot \lambda^2}}}
\end{aligned}$$

4.3 Wavelet Shrinkage

a.

$$f = (3, 4, 1, -2, 0, 4, -2, -4)^\top$$

We have

$$\begin{aligned} c_{j,k} &= \frac{1}{\sqrt{2}} \cdot (c_{j-1,2k} + c_{j-1,2k+1}) \\ d_{j,k} &= \frac{1}{\sqrt{2}} \cdot (c_{j-1,2k} - c_{j-1,2k+1}) \end{aligned}$$

and we start from $c_{0,k} = f_k$ for $k = 0, ..., N-1$ where $N = 8$.
Since f has 8 components, we have the scale $j = 1, 2, 3$:

$$\begin{aligned} c_{1,0} &= \frac{1}{\sqrt{2}} \cdot (3+4) = \frac{1}{\sqrt{2}} \cdot 7 \\ c_{1,1} &= \frac{1}{\sqrt{2}} \cdot (1 + (-2)) = -\frac{1}{\sqrt{2}} \\ c_{1,2} &= \frac{1}{\sqrt{2}} \cdot (0+4) = 2\sqrt{2} \\ c_{1,3} &= \frac{1}{\sqrt{2}} \cdot (-2 + (-4)) = -3\sqrt{2} \\ c_{2,0} &= \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot 7 + \left(-\frac{1}{\sqrt{2}} \right) \right) = 3 \\ c_{2,1} &= \frac{1}{\sqrt{2}} \cdot (2\sqrt{2} + (-3\sqrt{2})) = -1 \\ c_{3,0} &= \frac{1}{\sqrt{2}} \cdot (3 + (-1)) = \sqrt{2} \end{aligned}$$

$$
\begin{aligned}
d_{1,0} &= \frac{1}{\sqrt{2}} \cdot (3-4) = -\frac{1}{\sqrt{2}} \\
d_{1,1} &= \frac{1}{\sqrt{2}} \cdot (1-(-2)) = 3 \cdot \frac{1}{\sqrt{2}} \\
d_{1,2} &= \frac{1}{\sqrt{2}} \cdot (0-4) = -2\sqrt{2} \\
d_{1,3} &= \frac{1}{\sqrt{2}} \cdot (-2-(-4)) = \sqrt{2} \\
d_{2,0} &= \frac{1}{\sqrt{2}} \cdot \left(\frac{1}{\sqrt{2}} \cdot 7 - \left(-\frac{1}{\sqrt{2}}\right)\right) = 4 \\
d_{2,1} &= \frac{1}{\sqrt{2}} \cdot (2\sqrt{2} - (-3\sqrt{2})) = 5 \\
d_{3,0} &= \frac{1}{\sqrt{2}} \cdot (3-(-1)) = -\frac{1}{\sqrt{2}}
\end{aligned}
$$

From that, I can state the transformed signal:

$$\Rightarrow \hat{f} = \left(\sqrt{2}, -\frac{1}{\sqrt{2}}, 4, 5, -\frac{1}{\sqrt{2}}, 3 \cdot \frac{1}{\sqrt{2}}, -2\sqrt{2}, \sqrt{2}\right)^\top$$

b. The transformed signal $\hat{f}$ is approximately:

$$\hat{f} = (1.4, -0.7, 4, 5, -0.7, 2.1, 2.8, 1.4)^\top$$

Since we want to remove three coefficients, the threshold parameter T has to be chosen s.t.

$$3 \cdot \frac{1}{\sqrt{2}} \leq T$$

Now I can state the signal after the hard wavelet shrinkage (without touching the scaling coefficient):

$$\hat{f}' = (\sqrt{2}, 0, 4, 5, 0, 0, -2\sqrt{2}, 0)^\top$$

c. Computing the backtransform of the signal $\hat{f}'$ looks like this:
We have

$$
\begin{aligned}
c_{j,2k} &= \frac{1}{\sqrt{2}}(c_{j+1,k} + d_{j+1,k}) \\
c_{j,2k+1} &= \frac{1}{\sqrt{2}}(c_{j+1,k} - d_{j+1,k})
\end{aligned}
$$

and use the scale $j = 2, 1, 0$:

$$\begin{aligned}
c_{2,0} &= \frac{1}{\sqrt{2}}\left(\sqrt{2} + \left(-\frac{1}{\sqrt{2}}\right)\right) = \frac{1}{2} \\
c_{2,1} &= \frac{1}{\sqrt{2}}\left(\sqrt{2} - \left(-\frac{1}{\sqrt{2}}\right)\right) = \frac{3}{2} \\
\\
c_{1,0} &= \frac{1}{\sqrt{2}}(3 + 4) = 7 \cdot \frac{1}{\sqrt{2}} \\
c_{1,1} &= \frac{1}{\sqrt{2}}(3 - 4) = -\frac{1}{\sqrt{2}} \\
c_{1,2} &= \frac{1}{\sqrt{2}}(-1 + 5) = 2 \cdot \sqrt{2} \\
c_{1,3} &= \frac{1}{\sqrt{2}}(-1 - 5) = -3 \cdot \sqrt{2} \\
\\
c_{0,0} &= \frac{1}{\sqrt{2}}\left(7 \cdot \frac{1}{\sqrt{2}} + 0\right) = \frac{7}{2} \\
c_{0,1} &= \frac{1}{\sqrt{2}}\left(7 \cdot \frac{1}{\sqrt{2}} - 0\right) = \frac{7}{2} \\
c_{0,2} &= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + 0\right) = -\frac{1}{2} \\
c_{0,3} &= \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}} - 0\right) = -\frac{1}{2} \\
c_{0,4} &= \frac{1}{\sqrt{2}}(2 \cdot \sqrt{2} + (-2 \cdot \sqrt{2})) = 0 \\
c_{0,5} &= \frac{1}{\sqrt{2}}(2 \cdot \sqrt{2} - (-2 \cdot \sqrt{2})) = 4 \\
c_{0,6} &= \frac{1}{\sqrt{2}}(-3 \cdot \sqrt{2} + 0) = -3 \\
c_{0,7} &= \frac{1}{\sqrt{2}}(-3 \cdot \sqrt{2} - 0) = -3
\end{aligned}$$

Since $f'_k = c_{0,k}$, the signal is the following:

$$f' = \left(\frac{7}{2}, \frac{7}{2}, -\frac{1}{2}, -\frac{1}{2}, 0, 4, -3, -3\right)^\top$$