Image Processing and Computer Vision

Assignment T1

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1.1 Signal-to-Noise Ratio

a. Let's see, if $\mathsf{SNR}(f,g) = 0$:

$$\begin{aligned}
\mathsf{SNR}(f,g) &= 0 \\
10 \cdot \log_{10} \frac{\sigma^2(g)}{\sigma^2(n)} &= 0 \\
\log_{10} \frac{\sigma^2(g)}{\sigma^2(n)} &= 0 \\
\log_{10} \sigma^2(g) - \log_{10} \sigma^2(n) &= 0 \\
\log_{10} \sigma^2(g) &= \log_{10} \sigma^2(n) \\
\sigma^2(g) &= \sigma^2(n)
\end{aligned}$$

So this is the case, if the noise doesn't affect the original image.

b. For the sake of readability, I define a function as follows:

$$short(x,y) = \sum_{i=1}^{M} \sum_{j=1}^{N} (x-y)^2$$

$$\begin{aligned}
\mathsf{SNR}(u,g) &= \mathsf{SNR}(f,g) + 10\mathsf{dB} \\
\Leftrightarrow 10 \cdot \log_{10} \frac{\frac{1}{MN} \cdot short(g_{i,j}, \mu)}{\frac{1}{MN} \cdot short(u_{i,j}, g_{i,j})} &= 10 \cdot \log_{10} \frac{\frac{1}{MN} \cdot short(g_{i,j}, \mu)}{\frac{1}{MN} \cdot short(f_{i,j}, g_{i,j})} + 10 \\
\Leftrightarrow \log_{10} \frac{short(g_{i,j}, \mu)}{short(u_{i,j}, g_{i,j})} & \\
- \log_{10} \frac{short(g_{i,j}, \mu)}{short(f_{i,j}, g_{i,j})} &= 1 \\
\Leftrightarrow \log_{10} \frac{\frac{\sum_{i=1}^{M}\sum_{j=1}^{N}(g_{i,j})^2}{\sum_{i=1}^{M}\sum_{j=1}^{N}(u_{i,j}-g_{i,j})^2}}{\frac{\sum_{i=1}^{M}\sum_{j=1}^{N}(g_{i,j})^2}{\sum_{i=1}^{M}\sum_{j=1}^{N}(f_{i,j}-g_{i,j})^2}} &= 1 \\
\Leftrightarrow \log_{10} \frac{\sum_{i=1}^{M}\sum_{j=1}^{N}(f_{i,j}-g_{i,j})^2}{\sum_{i=1}^{M}\sum_{j=1}^{N}(u_{i,j}-g_{i,j})^2} &= 1 \\
\Leftrightarrow \frac{\sum_{i=1}^{M}\sum_{j=1}^{N}(f_{i,j}-g_{i,j})^2}{\sum_{i=1}^{M}\sum_{j=1}^{N}(u_{i,j}-g_{i,j})^2} &= 10
\end{aligned}$$

$$
\begin{aligned}
\Leftrightarrow \sum_{i=1}^{M}\sum_{j=1}^{N}(f_{i,j}-g_{i,j})^2 &= 10\cdot\sum_{i=1}^{M}\sum_{j=1}^{N}(u_{i,j}-g_{i,j})^2 \\
\Leftrightarrow \frac{1}{MN}\cdot\sum_{i=1}^{M}\sum_{j=1}^{N}(f_{i,j}-g_{i,j})^2 &= 10\cdot\frac{1}{MN}\cdot\sum_{i=1}^{M}\sum_{j=1}^{N}(u_{i,j}-g_{i,j})^2 \\
\Leftrightarrow \sigma^2(n) &= 10\cdot\sigma^2(n')
\end{aligned}
$$

where $\sigma^2(n')$ is the noise variance of the filtered image.
Interpretation:
As one can see, playing with the SNR can have a huge impact on the noise variance. The noise variance of the filtered image is 10 times higher than the noise variance of the original image.

c.

- If the initial image is constant, the variance is also constant. For example:

$$
\begin{aligned}
g=5\Rightarrow\sigma^2(g) &= \frac{1}{MN}\sum_{i=1}^{M}\sum_{j=1}^{N}(g_{i,j}-\mu)^2 \\
\overset{\mu=0}{\Leftrightarrow}\sigma^2(g) &= \frac{1}{MN}\cdot 25\cdot M\cdot N \\
\sigma^2(g) &= 25=5^2=g^2
\end{aligned}
$$

  This means that the resulting image is the noise function with some additional weighting which is the constant image.

- Let's try to calculate $\mathsf{SNR}(g,g)$:

$$
\mathsf{SNR}(g,g)=10\cdot\log_{10}\frac{\frac{1}{MN}\cdot\sum_{i=1}^{M}\sum_{j=1}^{N}(g_{i,j}-\mu)^2}{\underbrace{\frac{1}{MN}\cdot\sum_{i=1}^{M}\sum_{j=1}^{N}(g_{i,j}-g_{i,j})^2}_{=0}}
$$

  One cannot calculate this, because of the division by 0.
  Since this SNR is not defined, one can interpret the result as follows:
  Measuring how much your original image is deteriorated in comparison to your original image doesn't make sense, because you compare an image with itself.

1.2 Discrete vs. Continuous Convolution

a.

$$
\begin{aligned}
f_i &= \begin{cases} \frac{1}{2} & (i \in \{0,1\}) \\ 0 & (\text{else}) \end{cases} \\
(f*f)_k &= \sum_i f_{k-i} \cdot f_i = \sum_{i=0}^{1} f_{k-i} \cdot f_i \\
&= \frac{1}{2} \cdot \sum_{i=0}^{1} f_{k-i} = \frac{1}{2} \cdot (f_k + f_{k-1}) = g \\
(g*f)_k &= \sum_i g_{k-i} \cdot f_i = \sum_{i=0}^{1} g_{k-i} \cdot f_i \\
&= \frac{1}{2} \cdot \sum_{i=0}^{1} g_{k-i} = \frac{1}{2} \cdot \sum_{i=0}^{1} \left(\frac{1}{2} \cdot (f_k + f_{k-1}) \right)_{k-i} \\
&= \frac{1}{4} \cdot ((f_k + f_{k-1}) + (f_{k-1} + f_{k-2})) = h \\
(h*f)_k &= \sum_i h_{k-i} \cdot f_i = \sum_{i=0}^{1} h_{k-i} \cdot f_i \\
&= \frac{1}{2} \sum_{i=0}^{1} h_{k-i} = \frac{1}{2} \cdot \frac{1}{4} \cdot \sum_{i=0}^{1} ((f_k + f_{k-1}) + (f_{k-1} + f_{k-2}))_{k-i} \\
&= \frac{1}{8} [(((f_k + f_{k-1}) + (f_{k-1} + f_{k-2})) + ((f_{k-1} + f_{k-2}) + (f_{k-2} + f_{k-3}))] \\
&= \frac{1}{8} ((f_k + f_{k-1}) + 2 \cdot (f_{k-1} + f_{k-2}) + (f_{k-2} + f_{k-3})) = l \\
(l*f)_k &= \sum_i l_{k-i} \cdot f_i = \frac{1}{2} \cdot \sum_{i=0}^{1} l_{k-i} \\
&= \frac{1}{2} \cdot \frac{1}{8} \cdot \sum_{i=0}^{1} ((f_k + f_{k-1}) + 2 \cdot (f_{k-1} + f_{k-2}) + (f_{k-2} + f_{k-3}))_{k-i} \\
&= \frac{1}{16} \cdot [(f_k + f_{k-1}) + 2 \cdot (f_{k-1} + f_{k-2}) + (f_{k-2} + f_{k-3}) \\
&\qquad + (f_{k-1} + f_{k-2}) + 2 \cdot (f_{k-2} + f_{k-3}) + (f_{k-3} + f_{k-4})] \\
&= \frac{1}{16} \cdot [(f_k + f_{k-1}) + 3 \cdot (f_{k-1} + fk - 2) \\
&\qquad + 3 \cdot (f_{k-2} + f_{k-3}) + (f_{k-3} + f_{k-4})]
\end{aligned}
$$

b. For this exercise, I have a strong evidence that my solution is not correct. Nevertheless I want to put it here because of a small chance that it's right anyway ;-)

$$
\begin{aligned}
(f * f)(x) &= \int_{-\infty}^{\infty} f(x-y) \cdot f(y) dy \\
&= \int_{-\infty}^{-2} f(x-y) \cdot f(y) dy + \int_{-2}^{0} f(x-y) \cdot f(y) dy \\
&\quad + \int_{0}^{2} f(x-y) \cdot f(y) dy + \int_{2}^{\infty} f(x-y) \cdot f(y) dy \\
&= \int_{-2}^{0} f(x-y) \cdot f(y) dy + \int_{0}^{2} f(x-y) \cdot f(y) dy \\
&= \int_{-2}^{-1} f(x-y) \cdot f(y) dy + \int_{-1}^{1} f(x-y) \cdot f(y) dy + \int_{1}^{2} f(x-y) \cdot f(y) dy \\
&= \int_{-1}^{1} f(x-y) \cdot f(y) dy = \frac{1}{2} \cdot \int_{-1}^{1} f(x-y) dy \\
&= \frac{1}{2} \cdot \int_{x-1}^{x+1} \frac{1}{2} dy = \frac{1}{2} \cdot \left[\frac{1}{2} y \right]_{x-1}^{x+1} \\
&= \frac{1}{2} \cdot \left(\frac{1}{2} \cdot (x+1) - \frac{1}{2} \cdot (x-1) \right) \\
&= \frac{1}{2} \cdot \left(\frac{1}{2}x + \frac{1}{2} - \frac{1}{2}x + \frac{1}{2} \right) \\
&= \frac{1}{2}
\end{aligned}
$$

This result is why I'm not sure whether it is right or wrong. If you would repeat this convolution step 3 times as the exercise requests it you would get the same result again.

1.3 Properties of the Discrete Convolution

a. To show: $(\alpha \cdot f + \beta \cdot g) * w = \alpha \cdot (f * w) + \beta \cdot (g * w) \quad \forall \alpha, \beta \in \mathbb{R}$

$$
\begin{aligned}
((\alpha \cdot f + \beta \cdot g) * w)_i &= \sum_{k=-\infty}^{\infty} (\alpha \cdot f + \beta \cdot g)_{i-k} \cdot w_k \\
&= \sum_{k=-\infty}^{\infty} (\alpha \cdot f \cdot w_{2k-i} + \beta \cdot g \cdot w_{2k-i})_{i-k} \\
&= \sum_{k=-\infty}^{\infty} ((\alpha \cdot f \cdot w_{2k-i})_{i-k} + (\beta \cdot g \cdot w_{2k-i})_{i-k}) \\
&= \sum_{k=-\infty}^{\infty} (\alpha \cdot (f \cdot w_{2k-i})_{i-k} + \beta \cdot (g \cdot w_{2k-i})_{i-k}) \\
&= \sum_{k=-\infty}^{\infty} \alpha \cdot (f \cdot w_{2k-i})_{i-k} + \sum_{k=-\infty}^{\infty} \beta \cdot (g \cdot w_{2k-i})_{i-k} \\
&= \alpha \cdot \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k + \beta \cdot \sum_{k=-\infty}^{\infty} g_{i-k} \cdot w_k \\
&= \alpha \cdot (f * w)_i + \beta \cdot (g * w)_i
\end{aligned}
$$

b. To show: $f * w = w * f$

$$
\begin{aligned}
(f * w)_i &= \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k = \sum_{k=-\infty}^{\infty} w_k \cdot f_{i-k} \\
&= \sum_{k=-\infty}^{\infty} w_{k-i} \cdot f_{-k} = \sum_{k=\infty}^{-\infty} w_{i-k} \cdot f_k \\
&\overset{(*)}{=} \sum_{k=-\infty}^{\infty} w_{i-k} \cdot f_k \\
&= (w * f)_i
\end{aligned}
$$

where $(*)$ is because of the symmetry of the sum.

c. Claim:

$$
e = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}
$$

Proof:

$$
(f * e)_i = \sum_{k=-\infty}^{\infty} f_{i-k} \cdot e_k = \underbrace{\sum_{-\infty}^{-1} f_{i-k} \cdot e_k}_{=0} + f_i \cdot 1 + \underbrace{\sum_{1}^{\infty} f_{i-k} \cdot e_k}_{=0} = f_i
$$

1.4 Continous Fourier Transform

a. We have the 1-D hat function:

$$f(x) = \begin{cases} \frac{2-|x|}{4} & (-2 \leq x \leq 2) \\ 0 & (\text{else}) \end{cases}$$

To make life easier, I re-define the function f as follows:

$$f(x) = \begin{cases} \frac{2+x}{4} & (-2 \leq x < 0) \\ \frac{2-x}{4} & (0 \leq x \leq 2) \\ 0 & (\text{else}) \end{cases}$$

Let's compute the Fourier transform:

$$\begin{aligned}
\hat{f}(u) &= \mathcal{F}[f](u) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i2\pi ux} dx \\
&= \int_{-2}^{2} f(x) \cdot e^{-i2\pi ux} dx \\
&= \int_{-2}^{0} f(x) \cdot e^{-i2\pi ux} dx + \int_{0}^{2} f(x) \cdot e^{-i2\pi ux} dx \\
&= \int_{-2}^{0} \frac{2+x}{4} \cdot e^{-i2\pi ux} dx + \int_{0}^{2} \frac{2-x}{4} \cdot e^{-i2\pi ux} dx \\
&= \left[\frac{2+x}{4} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot e^{-i2\pi ux}\right]_{-2}^{0} - \int_{-2}^{0} \frac{1}{4} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot e^{-i2\pi ux} dx \\
&\quad + \left[\frac{2-x}{4} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot e^{-i2\pi ux}\right]_{0}^{2} - \int_{0}^{2} \left(-\frac{1}{4}\right) \cdot \left(-\frac{1}{i2\pi u}\right) \cdot e^{-i2\pi ux} dx \\
&= \frac{1}{2} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot e^{0} - \frac{1}{4} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot \left[-\frac{1}{i2\pi u} \cdot e^{-i2\pi ux}\right]_{-2}^{0} \\
&\quad + \left(0 - \frac{1}{2} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot e^{0}\right) - \left(-\frac{1}{4}\right) \cdot \left(-\frac{1}{i2\pi u}\right) \cdot \left[-\frac{1}{i2\pi u} \cdot e^{-i2\pi ux}\right]_{0}^{2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2} \cdot \left(-\frac{1}{i2\pi u}\right) - \frac{1}{4} \cdot \left(-\frac{1}{i2\pi u}\right) \cdot \left[\left(-\frac{1}{i2\pi u}\right) - \left(-\frac{1}{i2\pi u}\right) \cdot e^{-i2\pi u(-2)}\right] \\
&\quad -\frac{1}{2} \cdot \left(-\frac{1}{i2\pi u}\right) - \left(-\frac{1}{4}\right) \cdot \left(-\frac{1}{-i2\pi u}\right) \cdot \left[-\frac{1}{i2\pi u} \cdot e^{-i2\pi u \cdot 2} - \left(-\frac{1}{i2\pi u}\right)\right] \\
&= -\frac{1}{2} \cdot \frac{1}{i2\pi u} - \frac{1}{4} \cdot \left(\frac{1}{i2\pi u}\right)^2 - \frac{1}{4} \cdot \left(\frac{1}{i2\pi u}\right)^2 \cdot e^{-i2\pi u \cdot (-2)} \\
&\quad +\frac{1}{2} \cdot \frac{1}{i2\pi u} + \frac{1}{4} \cdot \left(\frac{1}{i2\pi u}\right)^2 \cdot e^{-i2\pi u \cdot 2} - \frac{1}{4} \cdot \left(\frac{1}{i2\pi u}\right)^2 \\
&= -\frac{1}{2} \cdot \left(\frac{1}{i2\pi u}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{i2\pi u}\right)^2 \cdot \left(e^{-i2\pi u \cdot 2} - e^{i2\pi u \cdot 2}\right) \\
&= -\frac{1}{2} \cdot \frac{1}{i^2 4\pi^2 u^2} + \frac{1}{4} \cdot \frac{1}{i^2 4\pi^2 u^2} \cdot e^4 \left(e^{-i\pi u} - e^{i\pi u}\right) \\
&= -\frac{1}{2} \cdot \frac{1}{i^2 4\pi^2 u^2} + \frac{1}{4} \cdot \frac{1}{i^2 4\pi^2 u^2} \cdot e^4 \cdot (-2i \sin(\pi u)) \\
&= -\frac{1}{2} \cdot \frac{1}{i^2 4\pi^2 u^2} + \frac{1}{4} \cdot \frac{1}{-i2\pi^2 u^2} \cdot e^4 \cdot \sin(\pi u) \\
&= -\frac{1}{8i^2\pi^2 u^2} - \frac{e^4 \cdot \sin(\pi u)}{8i\pi^2 u^2} \\
&= \frac{-1 - e^4 \cdot \sin(\pi u) \cdot i}{8i^2\pi^2 u^2} \\
&= -\frac{1}{8i^2\pi^2 u^2} \cdot (1 + e^4 \cdot \sin(\pi u) \cdot i) \\
&\overset{i^2=-1}{=} \frac{1}{8\pi^2 u^2} \cdot (1 + e^4 \cdot \sin(\pi u) \cdot i)
\end{aligned}
$$

b.

$$
\begin{aligned}
|\hat{f}(u)| &= \left|\frac{1}{8\pi^2 u^2}\right| \cdot \left|1 + e^4 \cdot \sin(\pi u) \cdot i\right| \\
&= \frac{1}{8\pi^2 u^2} + \left|\frac{e^4 \cdot \sin(\pi u) \cdot i}{8\pi^2 u^2}\right| \\
&= \frac{1}{8\pi^2 u^2} + \left|\frac{e^4 \cdot i}{8\pi u}\right| \cdot \left|\frac{\sin(\pi u)}{\pi u}\right| \\
&= \frac{1}{8\pi^2 u^2} + \left|\frac{e^4 \cdot i}{8\pi u}\right| \cdot |\mathrm{sinc}(\pi u)|
\end{aligned}
$$