Image Processing and Computer Vision 2007/08
Example Solutions for Theoretical Assignments 5 (T5)

Problem 1 (Derivative Approximation)

(a) In order to determine the coefficients of the derivative mask, we have to perform a Taylor expansion for all points around the pixel f_i. This yields

$$\begin{aligned}
f_{i-2} &= f_i - \frac{2}{1}hf_i' + \frac{4}{2}h^2 f_i'' - \frac{8}{6}h^3 f_i''' + \frac{16}{24}h^4 f_i^{(4)} - \frac{32}{120}h^5 f_i^{(5)} + O(h^6)\\
f_{i-1} &= f_i - \frac{1}{1}hf_i' + \frac{1}{2}h^2 f_i'' - \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} - \frac{1}{120}h^5 f_i^{(5)} + O(h^6)\\
f_i &= f_i\\
f_{i+1} &= f_i + \frac{1}{1}hf_i' + \frac{1}{2}h^2 f_i'' + \frac{1}{6}h^3 f_i''' + \frac{1}{24}h^4 f_i^{(4)} + \frac{1}{120}h^5 f_i^{(5)} + O(h^6)\\
f_{i+2} &= f_i + \frac{2}{1}hf_i' + \frac{4}{2}h^2 f_i'' + \frac{8}{6}h^3 f_i''' + \frac{16}{24}h^4 f_i^{(4)} + \frac{32}{120}h^5 f_i^{(5)} + O(h^6)
\end{aligned}$$

Since we are interested in computing an approximation to the first derivative, we have to choose the parameters $\alpha_{-2}, \ldots, \alpha_2$ in such a way that the following holds:

$$\begin{aligned}
0f_i + 1f_i' + 0f_i'' + 0f_i''' + 0f_i^{(4)} &\overset{!}{=} \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2}\\
&\approx \quad (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2)f_i\\
&\quad +(-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2)hf_i'\\
&\quad +(4\alpha_{-2} + \alpha_{-1} + \alpha_1 + 4\alpha_2)\frac{1}{2}h^2 f_i''\\
&\quad +(-8\alpha_{-2} - \alpha_{-1} + \alpha_1 + 8\alpha_2)\frac{1}{6}h^3 f_i'''\\
&\quad +(16\alpha_{-2} + \alpha_{-1} + \alpha_1 + 16\alpha_2)\frac{1}{24}h^4 f_i^{(4)}\,.
\end{aligned}$$

This leads to the linear system

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1\\ -2 & -1 & 0 & 1 & 2\\ 4 & 1 & 0 & 1 & 4\\ -8 & -1 & 0 & 1 & 8\\ 16 & 1 & 0 & 1 & 16 \end{pmatrix} \begin{pmatrix} \alpha_{-2}\\ \alpha_{-1}\\ \alpha_0\\ \alpha_1\\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0\\ \frac{1}{h}\\ 0\\ 0\\ 0 \end{pmatrix}.$$

(b) Solving the previous system of equations gives

$$\begin{aligned}
\alpha_{-2} &= -\alpha_2 = \frac{1}{12h}\,, \\
\alpha_{-1} &= -\alpha_1 = \frac{-8}{12h}\,, \\
\alpha_0 &= 0\,.
\end{aligned}$$

Plugging the coefficients into the Taylor expansion gives

$$\begin{aligned}
&\frac{1}{12h}f_{i-2} - \frac{8}{12h}f_{i-1} + 0f_i + \frac{8}{12h}f_{i+1} - \frac{1}{12h}f_{i+2} \\
&= \underbrace{(1-8+0+8-1)}_{=0}\frac{1}{12}\frac{1}{h}f_i \\
&\quad + \underbrace{(-2+8+8-2)\frac{1}{12}\frac{h}{h}}_{=1}f_i' \\
&\quad + \underbrace{(4-8+8-4)}_{=0}\frac{1}{12}\frac{1}{2}\frac{h^2}{h}f_i'' \\
&\quad + \underbrace{(-8+8+8-8)}_{=0}\frac{1}{12}\frac{1}{6}\frac{h^3}{h}f_i''' \\
&\quad + \underbrace{(16-8+8-16)}_{=0}\frac{1}{12}\frac{1}{24}\frac{h^4}{h}f_i^{(4)} \\
&\quad + \underbrace{(-32+8+8-32)}_{=-48\neq 0}\frac{1}{12}\frac{1}{120}\frac{h^5}{h}f_i^{(5)} \\
&\quad + O(h^5) \\
&= f_i' - \frac{h^4}{30}f_i^{(5)} + O(h^5) \\
&= f_i' + O(h^4)\,,
\end{aligned}$$

which shows that the order of consistency of the approximation is 4.

(c) If a derivative of order d is approximated with n points, we have (w.l.o.g.) n coefficients $\{\alpha_0, \ldots, \alpha_{n-1}\}$ such that

$$\begin{aligned}
&\alpha_0 f_i + \alpha_1 f_{i+1} + \cdots + \alpha_{n-1} f_{i+(n-1)} \\
&\quad = \beta_0 f_i + \cdots + \beta_d h^d f_i^{(d)} + \cdots + \beta_{n-1} h^{n-1} f_i^{(n-1)} + \beta_n h^n f_i^{(n)} + \cdots,
\end{aligned}$$

where it is assumed that $d \leq n-1$. The values β_k $(k = 0, 1, \dots)$ are linear combinations of the coefficients α_i $(i = 0, \dots, n-1)$ (See previous a) and b)). Since we are approximating the $d-th$ derivative of f_i, we know that $\forall i$, $\alpha_i \sim \dfrac{1}{h^d}$. Thus, it must hold that $\beta_d = \dfrac{1}{h^d}$, and $\beta_k = \dfrac{0}{h^d}$, $\forall k \in \{0, \dots, n-1\} \setminus \{d\}$. The approximation error depends on the values β_k for $k \geq n$. In particular, the lower bound of the order of consistency is obtained when $\beta_n \sim \dfrac{1}{h^d} \neq 0$. Without loss of generality, let $\beta_k := \dfrac{\tilde{\beta}_k}{h^d}$ for $k \geq n$. In this case, the approximation reads

$$\begin{aligned}
&\alpha_0 f_i + \alpha_1 f_{i+1} + \cdots + \alpha_{n-1} f_{i+(n-1)} \\
&\quad = \quad f_i^{(d)} + \tilde{\beta}_n \frac{h^n}{h^d} f_i^{(n)} + \tilde{\beta}_{n+1} \frac{h^{n+1}}{h^d} f_i^{(n+1)} + \dots \\
&\quad = \quad f_i^{(d)} + O(h^{n-d}),
\end{aligned}$$

which shows that the order of consistency is at least $n - d$.

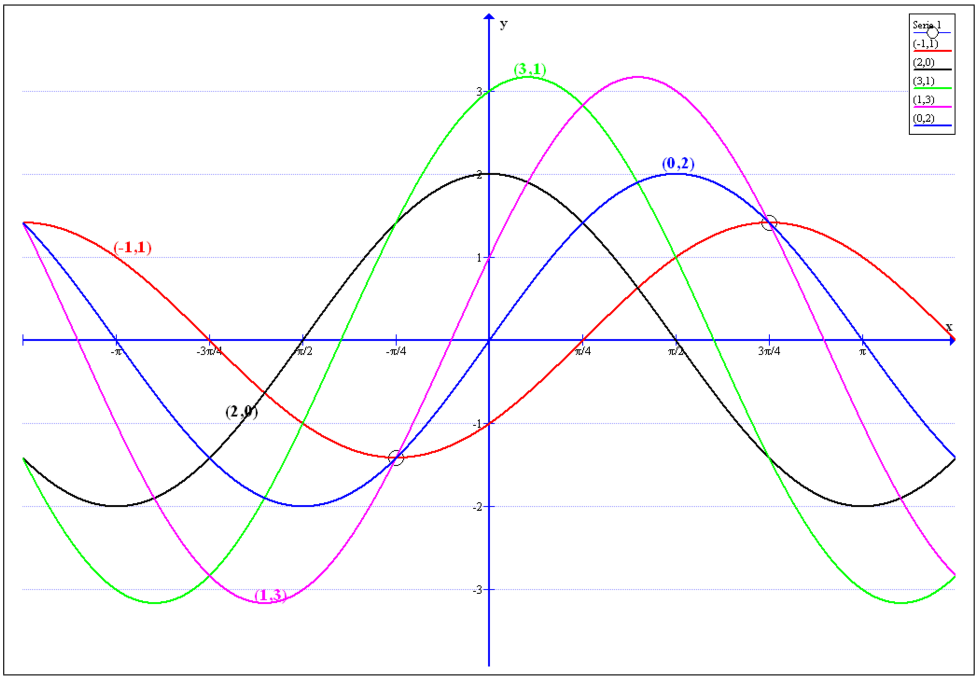

Figure 1: Plot of the five curves corresponing to the five given points. Note the intersection points at $\left(3\pi/4, \sqrt{2}\right)^\top$ and $\left(-\pi/4, -\sqrt{2}\right)^\top$.

Problem 2 (Hough Transform)

(a) As described in the lecture, what has to be done is as follows: For each given point $(x_i, y_i)^\top$, determine all lines that pass through this point. For this porpose, the equation

$$x_i \cos\phi + y_i \sin\phi - d = 0 \tag{1}$$

was given, which describes an affine plane in the 2-dimensional space by its normal vector $(\cos\phi, \sin\phi)^\top$ and its distance $d \in \mathbb{R}$ to the origin of the coordinate system. As there exists for each given point a line for any angle $\phi \in [0, \pi]$, equation (1) can be rewritten in its parametrised form as

$$d(\phi) = x_i \cos\phi + y_i \sin\phi \tag{2}$$

Now we are in the position to plot all parametrised curves for all given points, see figure 1. Drawing the curves for all points side by side, one has to realise what it means if two curves intersect in some point $(\alpha, d)^\top$: First of all this means that the two points that created the two curves share a line, i.e. there is a line that goes through both points. It is characterised by the position of the intersection point: the first component α describes the angular offset of the normal vector (from the x-axis, counterclockwise), the second component d specifies the distance from the coordinate origin (in direction of the normal vector). Regarding figure 1, we can now interpret a lot of details. Each intersection of two trigonometric curves coincides with a line passing through multiple points. If more than two curves intersect in one point, this implies that more than two points lie on the corresponding line. With our given point set, this scenario occurs only once for the line with normal vector $\left(\cos\frac{3\pi}{4}, \sin\frac{3\pi}{4}\right)^\top = \left(-\sqrt{2}, \sqrt{2}\right)^\top$ and distance $\sqrt{2}$ from the origin. The second visible intersection point at $\left(-\pi/4, -\sqrt{2}\right)^\top$ describes the same line, even if the normal vector and distance are not the same: the normal vector has inverted direction: $\left(\cos\frac{-\pi}{4}, \sin\frac{-\pi}{4}\right)^\top = \left(\sqrt{2}, -\sqrt{2}\right)^\top$. Since the distance of this second intersection is negated as well, the corrsponding line coincides with the first line. This behaviour also fits to the theory perfectly: rotating a line by 180° should not change it (note: $\frac{3\pi}{4} - \frac{-\pi}{4} = \pi$). Figure shows the result of our calculations. The described line goes through 3 points.

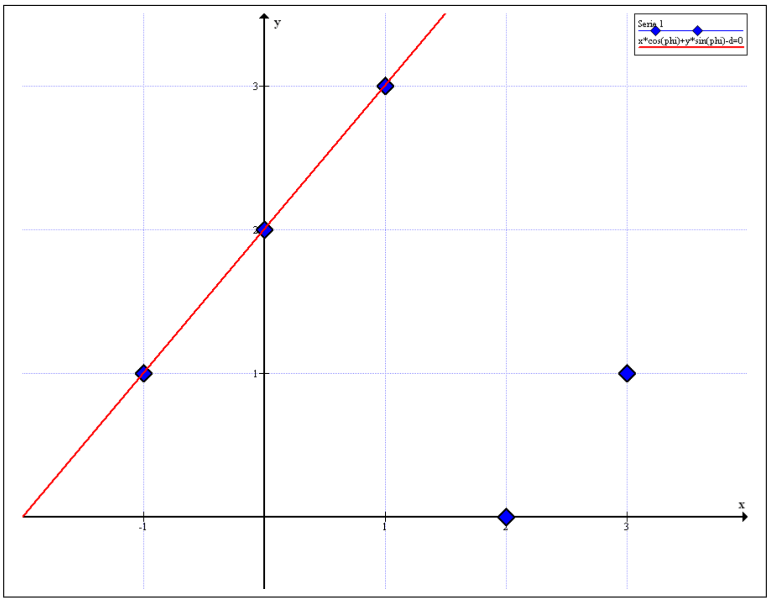

Figure 2: Given point set together with found line.

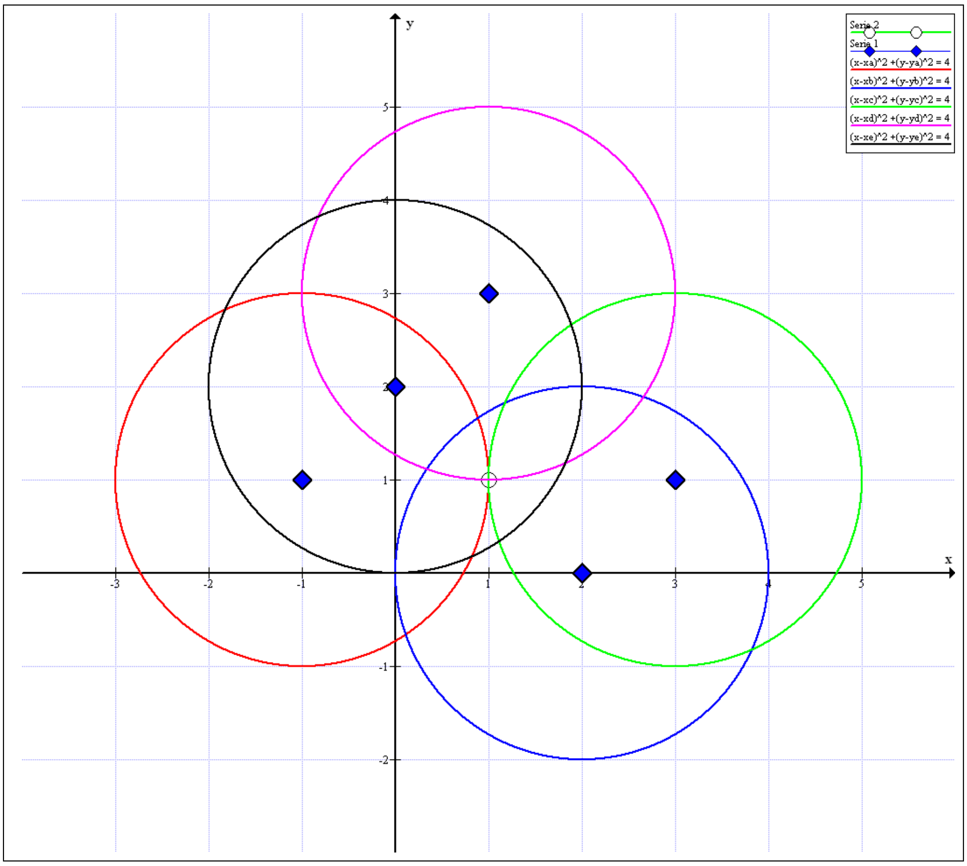

Figure 3: Circles belonging to the points. Note the intersection of three of those circles in point $(1,1)^\top$.

(b) In the case of matching circles, the problem size increases drastically. But if we restrict ourselves to circles of radius 2, things get simpler. In part (a) we considered lines parametrised by the angle ϕ of their normal vector:

$$d(\phi) = x_i \cos\phi + y_i \sin\phi$$

So, for every possible angle ϕ, there existed a suitable line. Now, in the case of circles, we have to recognise that for a given point $(x_i, y_i)^\top$, for every possible point $(a,b)^\top \in \mathbb{R}^2$ there exists a circle around that point that touches the point $(x_i, y_i)^\top$. This circle follows the equation

$$(x_i - a)^2 + (y_i - b)^2 - r^2 = 0 \tag{3}$$

Bringing r to the right hand side and taking the square root, one could end up with a two dimensional parametrisation of all relevant circles:

$$r(a,b) = \sqrt{(x_i - a)^2 + (y_i - b)^2} \tag{4}$$

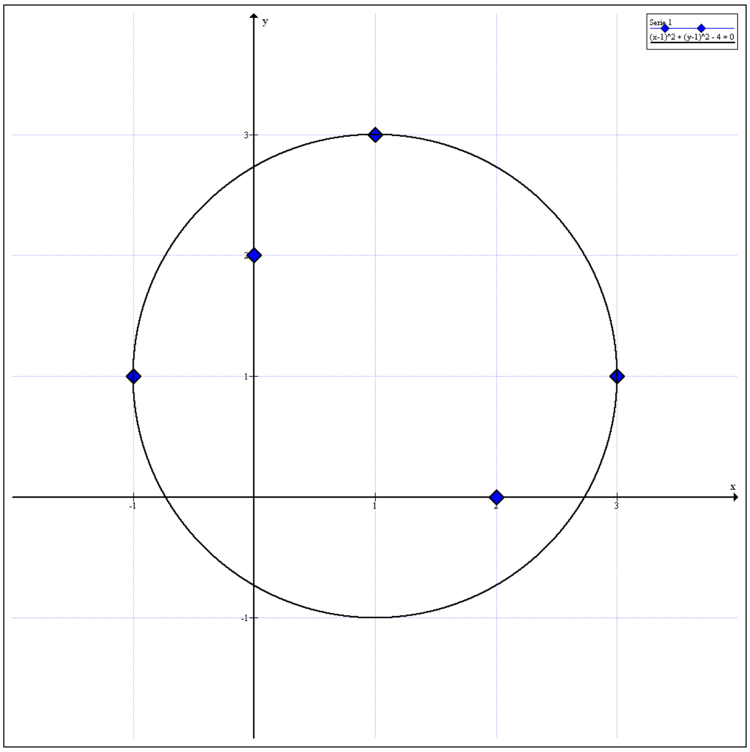

Figure 4: Given points together with the found circle. Its centre is $(1, 1)^\top$ and the radius is 2.

Here, r describes the radius of the circle. But since we shall just focus on circles with radius $r = 2$ in this exercise, we have

$$(x_i - a)^2 + (y_i - b)^2 = 4 \tag{5}$$

Since $(x_i, y_i)^\top$ are fixed for each point, a and b are the new variable parameters in this equation and since $(x_i - a)^2 = (a - x_i)^2$ what we get is nothing else than a new equation of a circle that contains all the centre points $(a, b)^\top$ of suitable circles with radius 2:

$$(a - x_i)^2 + (b - y_i)^2 - 4 = 0 \tag{6}$$

Figure 3 visualises these circles for all the given points: in $(1, 1)^\top$, the three circles asociated with the points $(3, 1)^\top$, $(-1, 1)^\top$ and $(1, 3)^\top$ intersect.

Thus, these three points lie on the circle with centre $(1, 1)^\top$ and radius 2, a plot of this circle can be seen in figure 4.

Problem 3 (Cooccurence Matrices)

As $d = (-1, -1)^T$ and the x-axis points to the right and the y-axis points downwards, we have to compare each pixel with its upper left neighbour.

j	
	i

By computing the relative frequency of each combination of greyvalue pairs (i, j) we obtain the following cooccurrence matrix C_{ij}:

$$\frac{1}{30}\begin{array}{|c|c|c|c|}\hline 1 & 2 & 1 & 1 \\ \hline 2 & 5 & 1 & 1 \\ \hline 2 & 2 & 3 & 2 \\ \hline 1 & 2 & 2 & 2 \\ \hline\end{array}$$