

Problem 1 (Continuous Nonquadratic Variational Methods)

(a) For an energy functional of the form

$$E(u) = \int_b^a F(u, u_x)$$

the associated Euler-Lagrange equation is given by

$$F_u - \frac{\partial}{\partial x} F_{u_x} = 0 .$$

In our case, the corresponding derivatives read

$$\begin{aligned} F_u &= u - f , \\ F_{u_x} &= \alpha \lambda^2 \frac{1}{2 \sqrt{1 + u_x^2/\lambda^2}} \frac{2 u_x}{\lambda^2} = \alpha \left(\frac{1}{\sqrt{1 + u_x^2/\lambda^2}} u_x \right) . \end{aligned}$$

Taking the derivative with respect to x of the second term yields

$$\frac{\partial}{\partial x} F_{u_x} = \alpha \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{1 + u_x^2/\lambda^2}} u_x \right) .$$

If desired this can be further simplified to

$$\begin{aligned} \alpha \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{1 + u_x^2/\lambda^2}} u_x \right) &= \alpha \left(\frac{u_{xx} \sqrt{1 + u_x^2/\lambda^2} - u_x \frac{1}{2 \sqrt{1 + u_x^2/\lambda^2}} \frac{2 u_x}{\lambda^2} u_{xx}}{1 + u_x^2/\lambda^2} \right) \\ &= \alpha \left(\frac{\frac{u_{xx}}{\sqrt{1 + u_x^2/\lambda^2}} (1 + u_x^2/\lambda^2) - \frac{u_{xx}}{\sqrt{1 + u_x^2/\lambda^2}} \frac{(u_x)^2}{\lambda^2}}{1 + u_x^2/\lambda^2} \right) \\ &= \alpha \frac{u_{xx}}{(1 + u_x^2/\lambda^2)^{\frac{3}{2}}} . \end{aligned}$$

Putting everything together, we obtain the Euler-Lagrange equation

$$u - f - \alpha \frac{\partial}{\partial x} \left(\frac{1}{\sqrt{1 + u_x^2/\lambda^2}} u_x \right) = 0$$

with natural boundary conditions $F_{u_x}(a) = F_{u_x}(b) = 0$.

- (b) The value of the parameter λ steers the preservations of discontinuities in the solution. This can be seen from the Euler-Lagrange equation, where the smoothness term results in the discontinuity-preserving diffusion process

$$\partial_t u = \frac{\partial}{\partial x} \left(\underbrace{\frac{1}{\sqrt{1 + u_x^2/\lambda^2}}}_{g(u_x^2)} u_x \right).$$

For very large values of λ , the diffusivity $g(u_x^2)$ approaches 1 which comes down to linear diffusion

$$\partial_t u = \frac{\partial}{\partial x} (u_x) = u_{xx}.$$

Thus, no discontinuities are preserved. For small values of λ , however, the diffusivity $g(u_x^2)$ depends mainly on the argument u_x . For large gradients u_x it even tends towards 0. This in turn inhibits smoothness at discontinuities in the solution.

- (c) It is sufficient to prove that $E(u)$ is strictly convex in u , as it follows that $E(u)$ has a single global minimum which is the unique solution of the Euler-Lagrange equation.

First we show the convexity of the data term of our functional, which is given by

$$\frac{1}{2} (u - f)^2 =: D(u).$$

We exploit the fact that a function is strictly convex if its second derivative is positive. Thus we compute

$$\frac{\partial^2}{\partial u^2} \left(\frac{1}{2} (u - f)^2 \right) = \frac{\partial}{\partial u} (u - f) = 1 > 0$$

which shows the strict convexity in u . Now we consider the smoothness term, given by

$$\alpha \lambda^2 \sqrt{1 + u_x^2/\lambda^2} =: S(u_x).$$

In part (a) we already computed the first derivative

$$\frac{\partial}{\partial u_x} S(u_x) = \frac{\partial}{\partial u_x} \left(\alpha \lambda^2 \sqrt{1 + u_x^2/\lambda^2} \right) = \frac{\alpha u_x}{\sqrt{1 + u_x^2/\lambda^2}}.$$

The second derivative is given by

$$\begin{aligned} \frac{\partial^2}{\partial u_x^2} S(u_x) &= \frac{\partial}{\partial u_x} \left(\frac{\alpha u_x}{\sqrt{1 + u_x^2/\lambda^2}} \right) = \alpha \left(\frac{\sqrt{1 + u_x^2/\lambda^2} - u_x \frac{u_x}{\lambda^2 \sqrt{1 + u_x^2/\lambda^2}}}{1 + u_x^2/\lambda^2} \right) \\ &= \alpha \frac{1}{(1 + u_x^2/\lambda^2)^{3/2}} \end{aligned}$$

As $\alpha > 0$ and $1 + u_x^2/\lambda^2 \geq 1$, the second derivative is positive, thus also our smoothness term is strictly convex in u .

We now use these results to show the strict convexity of our energy functional. A functional $E : X \rightarrow Y$ is strictly convex if it holds $\forall u, v \in X \forall \beta \in]0, 1[$:

$$E(\beta u + (1 - \beta) v) < \beta E(u) + (1 - \beta) E(v)$$

In our case, we have

$$\begin{aligned} &E(\beta u + (1 - \beta) v) \\ &= \int_a^b D(\beta u + (1 - \beta) v) + S((\beta u + (1 - \beta) v)_x) dx \\ &= \int_a^b D(\beta u + (1 - \beta) v) dx + \int_a^b S((\beta u + (1 - \beta) v)_x) dx \\ &= \int_a^b D(\beta u + (1 - \beta) v) dx + \int_a^b S(\beta u_x + (1 - \beta) v_x) dx \end{aligned} \tag{1}$$

where in the last step we have used the linearity of the differential operator. Due to the convexity of D we have

$$D(\beta u + (1 - \beta) v) < \beta D(u) + (1 - \beta) D(v)$$

As $a \leq b$, it follows that

$$\int_a^b D(\beta u + (1 - \beta) v) dx < \int_a^b \beta D(u) + (1 - \beta) D(v) dx \tag{2}$$

Analogously we want to derive an inequality for the second term. As S is strictly convex, it holds that

$$S(\beta u_x + (1 - \beta) v_x) < \beta S(u_x) + (1 - \beta) S(v_x)$$

Again we use the fact that $a \leq b$, which leads us to

$$\int_a^b S(\beta u_x + (1 - \beta) v_x) dx < \int_a^b \beta S(u_x) + (1 - \beta) S(v_x) dx \quad (3)$$

Finally, we combine equation (1) with inequalities (2) and (3)

$$\begin{aligned} & E(\beta u + (1 - \beta) v) \\ &= \int_a^b D(\beta u + (1 - \beta) v) + S((\beta u + (1 - \beta) v)_x) dx \\ &= \int_a^b D(\beta u + (1 - \beta) v) dx + \int_a^b S((\beta u + (1 - \beta) v)_x) dx \\ &< \int_a^b \beta D(u) + (1 - \beta) D(v) dx + \int_a^b \beta S(u_x) + (1 - \beta) S(v_x) dx \\ &= \beta \int_a^b D(u) + S(u_x) dx + (1 - \beta) \int_a^b D(v) + S(v_x) dx \\ &= \beta E(u) + (1 - \beta) E(v) \end{aligned}$$

which concludes the proof.

Problem 2 (Discrete Variational Methods)

- (a) In analogy to the functional considered in Problem 1, we write down a discrete version of $E(u)$ as follows:

$$E(u) := \frac{1}{2} \sum_{k=1}^N (u_k - f_k)^2 + \alpha \sum_{k=1}^{N-1} \lambda^2 \sqrt{1 + \frac{(u_{k+1} - u_k)^2}{\lambda^2 h^2}}.$$

Here, we assume that the finite forward difference and the length of the signal f , i.e., $N = \frac{b-a}{h}$, depend on the pixel distance $h > 0$, which is often set to 1 in practice.

- (b) The minimiser of the discrete functional $E(u)$ necessarily satisfies the nonlinear system of equations $\frac{\partial E(u)}{\partial u_k} = 0$ for all $k = 1, \dots, N$. Thus, we have to calculate partial derivatives distinguishing boundary pixels from inner pixels:

$$\frac{\partial E(u)}{\partial u_1} = u_1 - f_1 - \frac{\alpha}{h^2} \frac{u_2 - u_1}{\sqrt{1 + \frac{(u_2 - u_1)^2}{\lambda^2 h^2}}}, \quad (\text{for } k = 1),$$

$$\frac{\partial E(u)}{\partial u_k} = u_k - f_k - \frac{\alpha}{h^2} \frac{u_{k+1} - u_k}{\sqrt{1 + \frac{(u_{k+1} - u_k)^2}{\lambda^2 h^2}}} + \frac{\alpha}{h^2} \frac{u_k - u_{k-1}}{\sqrt{1 + \frac{(u_k - u_{k-1})^2}{\lambda^2 h^2}}}, \quad (\text{for } k = 2, \dots, N-1),$$

$$\frac{\partial E(u)}{\partial u_N} = u_N - f_N + \frac{\alpha}{h^2} \frac{u_N - u_{N-1}}{\sqrt{1 + \frac{(u_N - u_{N-1})^2}{\lambda^2 h^2}}}, \quad (\text{for } k = N).$$

Problem 3 (Wavelet Shrinkage)

- (a) We first compute the Fast Wavelet Transform (Lecture 6, slide 15) from the original signal $f = (3, 4, 1, -2, 0, 4, -2, -4)$ via

Level 1:

$$\begin{aligned}c_{1,0} &= \frac{1}{\sqrt{2}}(c_{0,0} + c_{0,1}) \\ &= \frac{1}{\sqrt{2}}(3 + 4) = \frac{7}{\sqrt{2}}\end{aligned}$$

$$\begin{aligned}d_{1,0} &= \frac{1}{\sqrt{2}}(c_{0,0} - c_{0,1}) \\ &= \frac{1}{\sqrt{2}}(3 - 4) = \frac{-1}{\sqrt{2}}\end{aligned}$$

$$c_{1,1} = \frac{1}{\sqrt{2}}(1 + (-2)) = \frac{-1}{\sqrt{2}}$$

$$d_{1,1} = \frac{1}{\sqrt{2}}(1 - (-2)) = \frac{3}{\sqrt{2}}$$

$$c_{1,2} = \frac{1}{\sqrt{2}}(0 + 4) = \frac{4}{\sqrt{2}}$$

$$d_{1,2} = \frac{1}{\sqrt{2}}(0 - 4) = \frac{-4}{\sqrt{2}}$$

$$c_{1,3} = \frac{1}{\sqrt{2}}((-2) + (-4)) = \frac{-6}{\sqrt{2}}$$

$$d_{1,3} = \frac{1}{\sqrt{2}}((-2) - (-4)) = \frac{2}{\sqrt{2}}$$

Level 2:

$$c_{2,0} = \frac{1}{\sqrt{2}}\left(\frac{7}{\sqrt{2}} + \frac{-1}{\sqrt{2}}\right) = 3$$

$$d_{2,0} = \frac{1}{\sqrt{2}}\left(\frac{7}{\sqrt{2}} - \frac{-1}{\sqrt{2}}\right) = 4$$

$$c_{2,1} = \frac{1}{\sqrt{2}}\left(\frac{4}{\sqrt{2}} + \frac{-6}{\sqrt{2}}\right) = -1$$

$$d_{2,1} = \frac{1}{\sqrt{2}}\left(\frac{4}{\sqrt{2}} - \frac{-6}{\sqrt{2}}\right) = 5$$

Level 3:

$$c_{3,0} = \frac{1}{\sqrt{2}}(3 + (-1)) = \frac{2}{\sqrt{2}}$$

$$d_{3,0} = \frac{1}{\sqrt{2}}(3 - (-1)) = \frac{4}{\sqrt{2}}$$

The transformed signal \hat{f} is now given by the following coefficients:

$$(c_{3,0}, d_{3,0}, d_{2,0}, d_{2,1}, d_{1,0}, d_{1,1}, d_{1,2}, d_{1,3})$$

which corresponds to

$$\hat{f} = \left(\frac{2}{\sqrt{2}}, \frac{4}{\sqrt{2}}, 4, 5, \frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, \frac{-4}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right).$$

- (b) Looking at the absolute values of our transformed signal \hat{f} , we see that there are three wavelet coefficients that are less or equal to $\frac{3}{\sqrt{2}}$ (remember that we do NOT alter the scaling coefficient, i.e. the first entry remains unaltered). As we wanted to remove three coefficients, we can take this value as our threshold T (the next larger value is $\frac{4}{\sqrt{2}}$, so every threshold T with $\frac{3}{\sqrt{2}} \leq T < \frac{4}{\sqrt{2}}$ would do the same job). The modified signal is then given by

$$\hat{f}' = \left(\frac{2}{\sqrt{2}}, \frac{4}{\sqrt{2}}, 4, 5, 0, 0, \frac{-4}{\sqrt{2}}, 0 \right).$$

Note that all removed coefficients ($d_{1,0}$, $d_{1,1}$ and $d_{1,3}$) represent high frequencies. They correspond to the first, second and fourth quarter of the original signal f .

(c) We now compute the backtransform of our modified signal \widehat{f}' :

Level 2:

$$\begin{aligned} c'_{2,0} &= \frac{1}{\sqrt{2}} (c'_{3,0} + d'_{3,0}) \\ &= \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} \right) = 3 \\ c'_{2,1} &= \frac{1}{\sqrt{2}} (c'_{3,0} - d'_{3,0}) \\ &= \frac{1}{\sqrt{2}} \left(\frac{2}{\sqrt{2}} - \frac{4}{\sqrt{2}} \right) = -1 \end{aligned}$$

Level 1:

$$\begin{aligned} c'_{1,0} &= \frac{1}{\sqrt{2}} (3 + 4) = \frac{7}{\sqrt{2}} \\ c'_{1,1} &= \frac{1}{\sqrt{2}} (3 - 4) = \frac{-1}{\sqrt{2}} \\ c'_{1,2} &= \frac{1}{\sqrt{2}} ((-1) + 5) = \frac{4}{\sqrt{2}} \\ c'_{1,3} &= \frac{1}{\sqrt{2}} ((-1) - 5) = \frac{-6}{\sqrt{2}} \end{aligned}$$

Level 0 (backtransformed signal):

$$\begin{aligned} c'_{0,0} &= \frac{1}{\sqrt{2}} \left(\frac{7}{\sqrt{2}} + 0 \right) = 3.5 \\ c'_{0,1} &= \frac{1}{\sqrt{2}} \left(\frac{7}{\sqrt{2}} - 0 \right) = 3.5 \\ c'_{0,2} &= \frac{1}{\sqrt{2}} \left(\frac{-1}{\sqrt{2}} + 0 \right) = -0.5 \\ c'_{0,3} &= \frac{1}{\sqrt{2}} \left(\frac{-1}{\sqrt{2}} - 0 \right) = -0.5 \\ c'_{0,4} &= \frac{1}{\sqrt{2}} \left(\frac{4}{\sqrt{2}} + \frac{-4}{\sqrt{2}} \right) = 0 \\ c'_{0,5} &= \frac{1}{\sqrt{2}} \left(\frac{4}{\sqrt{2}} - \frac{-4}{\sqrt{2}} \right) = 4 \\ c'_{0,6} &= \frac{1}{\sqrt{2}} \left(\frac{-6}{\sqrt{2}} + 0 \right) = -3 \\ c'_{0,7} &= \frac{1}{\sqrt{2}} \left(\frac{-6}{\sqrt{2}} - 0 \right) = -3 \end{aligned}$$

Our modified backtransformed signal is now given by

$$f' = c'_0 = (3.5, 3.5, -0.5, -0.5, 0, 4, -3, -3) .$$

Not surprisingly, the signal has been smoothed. Note that this smoothing is local: the coefficients f_4 and f_5 have not been altered. This is due

to the local nature of Haar wavelets. As we only modified the wavelet coefficients that correspond to the first, second and fourth quarter of the original signal, it makes perfect sense that the third quarter remains unaltered.