Image Processing and Computer Vision 2007/08

Example Solutions for Theoretical Assignments 2 (T2)

Problem 1 (Image Pyramids)

The *Gaussian pyramid* $\left\{v^N, ..., v^0\right\}$ of a signal $u = (u_0, ..., u_{2^N})^T$ is defined as

$$\begin{array}{rcl} v^N & := & u, \\ v^{k-1} & := & R_k^{k-1} v^k \end{array}$$

for $k = N, \ldots, 1$ with

$$R_k^{k-1} := \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \ldots & 0 & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

for $k \geq 2$ and $R_1^0 := \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

Applied to our signal $u := (-20, -2, 11, -4, -12, -20, 6, 24, 12)^T$ we get

$$\begin{array}{rcl} v^3 & = & u \\ v^2 & = & R_3^2 v^3 \\ & = & \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -20 \\ -2 \\ 11 \\ -4 \\ -12 \\ -20 \\ 6 \\ 24 \\ 12 \end{pmatrix} \\ & = & (-14, 4, -12, 4, 16)^T \\ v^1 & = & R_2^1 v^2 \\ & = & \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} \\ & = & (-8, -4, 12)^\top \end{array}$$

$$
\begin{aligned}
v^0 &= R_1^0 v^1 \\
&= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
&= 0
\end{aligned}
$$

The *Laplacian pyramid* $\left\{w^N, \ldots, w^0\right\}$ of a signal $u = (u_0, \ldots, u_{2^N})^\top$ with Gaussian pyramid $\left\{v^N, \ldots, v^0\right\}$ is defined as

$$
\begin{aligned}
w^0 &:= v^0, \\
w^k &:= v^k - R_{k-1}^k v^k
\end{aligned}
$$

for $k = N, \ldots, 1$ with

$$
P_k^{k+1} := \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 & 0 \\
0 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & \ldots & 0 & 0 & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
0 & 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 & \ldots & 0 & 0 & 1
\end{pmatrix}
$$

for $k \geq 1$ and $P_0^1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Applied to our signal, we obtain

$$
\begin{aligned}
w^0 &= v^0 = 0 \\
w^1 &= v^1 - P_0^1 v^0 \\
&= \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} - 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
w^2 &= v^2 - P_1^2 v^1 \\
&= \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
&= (-6, 10, -8, 0, 4)^\top \\
w^3 &= v^3 - P_2^3 v^2 \\
&= \begin{pmatrix} -20 \\ -2 \\ 11 \\ -4 \\ -12 \\ -20 \\ 6 \\ 24 \\ 12 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} \\
&= (-6, 3, 7, 0, 0, -16, 2, 14, -4)^\top
\end{aligned}
$$

Let us now reconstruct the original signal from the Laplacian pyramid:

$$
\begin{aligned}
v^0 &= w^0 = 0 \\
v^1 &= w^1 + P_0^1 v^0 \\
&= \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
v^2 &= w^2 + P_1^2 v^1 \\
&= \begin{pmatrix} -6 \\ 10 \\ -8 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -8 \\ -4 \\ 12 \end{pmatrix} \\
&= (-14, 4, -12, 4, 16)^\top
\end{aligned}
$$

$$
\begin{aligned}
v^3 &= w^3 + P_2^3 v^2 \\
&= \begin{pmatrix} -6 \\ 3 \\ 7 \\ 0 \\ 0 \\ -16 \\ 2 \\ 14 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -14 \\ 4 \\ -12 \\ 4 \\ 16 \end{pmatrix} \\
&= (-20, -2, 11, -4, -12, -20, 6, 24, 12)^\top
\end{aligned}
$$

Problem 2 (Discrete Fourier Transform)

(a) As we are given four sample points, the DFT of our signals is given as:

$$\widehat{f_p} = \frac{1}{2} \cdot \sum_{m=0}^{3} f_m \cdot \exp\left(\frac{-i\pi pm}{2}\right) \tag{1}$$

Plugging in Euler's formula, we get:

$$\widehat{f_p} = \frac{1}{2} \cdot \sum_{m=0}^{3} f_m \cdot \left(\cos\left(\frac{-\pi pm}{2}\right) + i \cdot \sin\left(\frac{-\pi pm}{2}\right)\right) \tag{2}$$

Since sin and cos are both 2π-periodic, we only need to consider four cases to evaluate the exponential term:

$p \cdot m \mod 4$	0	1	2	3
$\cos\left(\frac{-\pi pm}{2}\right) + i \cdot \sin\left(\frac{-\pi pm}{2}\right)$	1	$-i$	-1	i

Now we can easily compute the DFT of our signals f_1 and f_2:

$$
\begin{aligned}
\widehat{f}_{1,0} &= 1/2 \cdot (6+8+2+4) \\
&= 10 \\
\widehat{f}_{1,1} &= 1/2 \cdot (6 \cdot 1 + 8 \cdot (-i) + 2 \cdot (-1) + 4 \cdot i) \\
&= 2 - 2i \\
\widehat{f}_{1,2} &= 1/2 \cdot (6 \cdot 1 + 8 \cdot (-1) + 2 \cdot 1 + 4 \cdot (-1)) \\
&= -2 \\
\widehat{f}_{1,3} &= 1/2 \cdot (6 \cdot 1 + 8 \cdot i + 2 \cdot (-1) + 4 \cdot (-i)) \\
&= 2 + 2i
\end{aligned}
$$

So $\widehat{f_1}$ is given as $(10,\ 2-2i,\ -2,\ 2+2i)$. As expected, $\widehat{f}_{1,1} = \overline{\widehat{f}}_{1,3}$, i.e. the conjugate complex symmetry holds. This has to be the case, since we used a real-valued input signal.

$$
\begin{aligned}
\widehat{f}_{2,0} &= 1/2 \cdot (4+6+8+2) \\
&= 10 \\
\widehat{f}_{2,1} &= 1/2 \cdot (4 \cdot 1 + 6 \cdot (-i) + 8 \cdot (-1) + 2 \cdot i) \\
&= -2 - 2i \\
\widehat{f}_{2,2} &= 1/2 \cdot (4 \cdot 1 + 6 \cdot (-1) + 8 \cdot 1 + 2 \cdot (-1)) \\
&= 2 \\
\widehat{f}_{2,3} &= 1/2 \cdot (4 \cdot 1 + 6 \cdot i + 8 \cdot (-1) + 2 \cdot (-i)) \\
&= -2 + 2i
\end{aligned}
$$

So $\widehat{f_2}$ is given as: $(10,\ -2-2i,\ 2,\ -2+2i)$. Again, $\widehat{f}_{2,1}$ is the complex conjugate of $\widehat{f}_{2,3}$. Evidently, the DFT of the original and the shifted signal are different.

Let now take a look at the spectra of the two signals. They are given as follows:

$$\begin{aligned}
|\widehat{f}_{1,0}| &= \sqrt{\mathrm{Re}^2(\widehat{f}_{1,0}) + \mathrm{Im}^2(\widehat{f}_{1,0})} \\
&= \sqrt{10^2} \\
&= 10 \\
|\widehat{f}_{1,1}| &= \sqrt{2^2 + (-2)^2} \\
&= \sqrt{8} \\
|\widehat{f}_{1,2}| &= \sqrt{(-2)^2} \\
&= 2 \\
|\widehat{f}_{1,3}| &= \sqrt{2^2 + 2^2} \\
&= \sqrt{8}
\end{aligned}$$

$$\begin{aligned}
|\widehat{f}_{2,0}| &= \sqrt{10^2} \\
&= 10 \\
|\widehat{f}_{2,1}| &= \sqrt{(-2)^2 + (-2)^2} \\
&= \sqrt{8} \\
|\widehat{f}_{2,2}| &= \sqrt{2^2} \\
&= 2 \\
|\widehat{f}_{2,3}| &= \sqrt{(-2)^2 + 2^2} \\
&= \sqrt{8}
\end{aligned}$$

Although the DFTs are different, the spectra of the two signals are identical. This is due to the shift theorem that tells us that a shift in the spatial domain results in a rotation in the fourier domain. This rotation, however, does not become visible in the Fourier spectrum, since $\left|exp(-\frac{i\,2\pi\,pm_0}{M})\right| = 1$ for all shifts m_0.

(b) The highest frequency in f_1 is $\widehat{f}_{1,2}$. Setting this frequency to zero yields the following signal $\widehat{f'}_1 = (10,\ 2-2i,\ 0,\ 2+2i)$. Computing the inverse DFT of this signal can be done by applying the usual DFT to the complex conjugate of this signal

$$
\begin{aligned}
f'_{1,0} &= 1/2 \cdot (10 + 2 + 2i + 0 + 2 - 2i) \\
&= 7 \\
f'_{1,1} &= 1/2 \cdot (10 + (2 + 2i) \cdot (-i) + 0 \cdot (-1) + (2 - 2i) \cdot i) \\
&= 7 \\
f'_{1,2} &= 1/2 \cdot (10 + (2 + 2i) \cdot (-1) + 0 \cdot 1 + (2 - 2i) \cdot (-1)) \\
&= 3 \\
f'_{1,3} &= 1/2 \cdot (10 + (2 + 2i) \cdot i + 0 \cdot (-1) + (2 - 2i) \cdot (-i)) \\
&= 3
\end{aligned}
$$

and computing the complex conjugate of the results (which changes nothing since f'_1 is real-valued anyway). We can see that our new signal $f'_1 = (7,\ 7,\ 3,\ 3)$ is smoother than the original one. This can be explained by the fact that details (high frequencies) have been removed.

Problem 3 (Discrete Fourier Transform)

We show that it holds for all $p, q \in \{0, ..., M-1\}$:

$$
\langle v_p, v_q \rangle = \begin{cases} 1, & \text{if } p = q \\ 0, & \text{else} \end{cases}
$$

First we consider $p, q \in \{0, ..., M-1\}, q = p$. We have

$$
\begin{aligned}
\langle v_p, v_q \rangle &= \sum_{m=0}^{M-1} \frac{1}{\sqrt{M}} exp\left(\frac{2\pi ipm}{M}\right) \overline{\frac{1}{\sqrt{M}} exp\left(\frac{2\pi iqm}{M}\right)} \\
&= \frac{1}{M} \sum_{m=0}^{M-1} exp\left(\frac{2\pi ipm}{M}\right) exp\left(-\frac{2\pi iqm}{M}\right) \\
&= \frac{1}{M} \sum_{m=0}^{M-1} exp\Big(\frac{2\pi i\overbrace{(p-q)}^{0}m}{M}\Big) \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \underbrace{exp\,(0)}_{1} \\
&= \frac{M}{M} \;=\; 1
\end{aligned}
$$

Now we consider $p, q \in \{0, ..., M-1\}, q \neq p$. It holds that

$$
\begin{aligned}
\langle v_p, v_q \rangle &= \frac{1}{M} \sum_{m=0}^{M-1} exp\left(\frac{2\pi i(p-q)m}{M}\right) \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \left(exp\left(\frac{2\pi i(p-q)}{M}\right)\right)^m \\
\text{geometric series formula} \quad &= \frac{1}{M} \frac{1-\left(exp\left(\frac{2\pi i(p-q)}{M}\right)\right)^M}{1-exp\left(\frac{2\pi i(p-q)}{M}\right)} \qquad \begin{array}{l} 0 \leq p, q \leq M \\ \Rightarrow \frac{p-q}{M} \notin \mathbb{Z} \\ \Rightarrow exp\left(\frac{2\pi(p-q)}{M}\right) \neq 1 \end{array} \\
&= \frac{1}{M} \frac{1-exp\left(\frac{2\pi i(p-q)M}{M}\right)}{1-exp\left(\frac{2\pi i(p-q)}{M}\right)} \\
&= \frac{1}{M} \frac{1-exp\Big(2\pi i(p-q)\Big)}{1-exp\left(\frac{2\pi i(p-q)}{M}\right)} \qquad \begin{array}{l} (p-q) \in \mathbb{Z} \\ \Rightarrow exp\left(2\pi\left(p-q\right)\right) = 1 \end{array} \\
&= \frac{1}{M} \frac{1-1}{1-exp\left(\frac{2\pi i(p-q)}{M}\right)} \\
&= 0
\end{aligned}
$$

As the set $\{v_o, ...v_{M-1}\}$ has cardinality M, it follows that it forms an orthonormal basis of the M-dimensional vector space $\mathbb{C}^M$ with respect to $\langle \cdot, \cdot \rangle$.