Image Processing and Computer Vision 2007/08
Example Solutions for Theoretical Assignments 1 (T1)

Problem 1 (Signal-to-Noise Ratio)

We first review the definition of the SNR:

$$\mathrm{SNR}(f,g) := 10\log_{10}\left(\frac{\sigma^2(g)}{\sigma^2(n)}\right)$$

where $\sigma^2(g)$ is the variance of the original image and $\sigma^2(n)$ is the variance of the additive noise.

(a) We have

$$\mathrm{SNR}(f,g) = 0 \quad \iff \quad \log_{10}\left(\frac{\sigma^2(g)}{\sigma^2(n)}\right) = 0 \quad \iff \quad \sigma^2(g) = \sigma^2(n) \ .$$

That means in the case $\mathrm{SNR}(f,g) = 0$ the noise has the same variance as the image data.

(b) Here we use the fact that

$$\log_{10}\left(\frac{a}{b}\right) = \log_{10}(a) - \log_{10}(b) \ .$$

Let us assume that u is a restored version of the noisy image f such that $\mathrm{SNR}(u,g) = \mathrm{SNR}(f,g) + 10$. Let us write $f = g + n_1$ and $u = g + n_2$ for the additive noise on f and u. We calculate that

$$\begin{aligned}\mathrm{SNR}(u,g) = \mathrm{SNR}(f,g)+10 &\iff \log_{10}\left(\frac{\sigma^2(g)}{\sigma^2(n_2)}\right) = \log_{10}\left(\frac{\sigma^2(g)}{\sigma^2(n_1)}\right) + 1 \\ &\iff \log_{10}(\sigma^2(n_2)) = \log_{10}(\sigma^2(n_1)) - 1 \\ &\iff \sigma^2(n_2) = \frac{\sigma^2(n_1)}{10}\end{aligned}$$

That means the filtering algorithm reduced the variance of the noise by a factor 10.

(c) We have already seen that $\mathrm{SNR}(f,g) = 10\log_{10}(\sigma^2(g)) - 10\log_{10}(\sigma^2(n))$.

Case 1. If the initial image g is constant, we have $\sigma^2(g) = 0$, and then the first logarithm is not defined. In the limit we have $\lim_{x\to 0^+}\log_{10}(x) = -\infty$. That means for a constant image g and a non-constant noisy version f we would obtain $\mathrm{SNR}(f, g) = -\infty$. In this case the noise variance is infinitely many times as large as the image variance.

Case 2. If we try to calculate SNR(g, g), the variance of the noise is zero: $\sigma^2(n) = 0$. For a non-constant image this would lead to an SNR of $+\infty$ which expresses the fact that the quality cannot become better.

Problem 2 (Discrete vs. Continuous Convolution)

(a) We observe that with the given convolution kernel f one obtains for any discrete signal $g = (g_i)_{i \in \mathbb{Z}}$ the result

$$(g * f)_i = \sum_{j=-\infty}^{\infty} f_{i-j}\, g_j = \frac{1}{2}\, g_i + \frac{1}{2}\, g_{i-1}\ .$$

Applying this iteratively leads to

$$(f * f)_i = \begin{cases} 0, & i < 0 \text{ or } i > 2, \\ \frac{1}{4}, & i = 0, 2, \\ \frac{2}{4}, & i = 1 \end{cases}$$

$$(f * f * f)_i = \begin{cases} 0, & i < 0 \text{ or } i > 3, \\ \frac{1}{8}, & i = 0, 3, \\ \frac{3}{8}, & i = 1, 2 \end{cases}$$

$$(f * f * f * f)_i = \begin{cases} 0, & i < 0 \text{ or } i > 4, \\ \frac{1}{16}, & i = 0, 4, \\ \frac{4}{16}, & i = 1, 3, \\ \frac{6}{16}, & i = 2\ . \end{cases}$$

(b) For the given convolution kernel f and any continuous signal $g(x)$ we have

$$(g * f)(x) = \int_{-\infty}^{\infty} f(x - z)\, g(z)\, \mathrm{d}z \ = \int_{x-1}^{x+1} \frac{1}{2}\, g(z)\, \mathrm{d}z\ .$$

For $f * f$ we therefore obtain

$$(f * f)(x) = \int\limits_{x-1}^{x+1} \frac{1}{2}\, f(z)\, \mathrm{d}z$$

$$= \begin{cases} 0, & x \leq -2, \\ \int_{-1}^{x+1} \frac{1}{4}\, \mathrm{d}z, & -2 < x \leq 0, \\ \int_{x-1}^{1} \frac{1}{4}\, \mathrm{d}z, & 0 < x \leq 2, \\ 0, & x > 2 \end{cases}$$

and thus

$$(f * f)(x) = \begin{cases} 0, & x \leq -2, \\ \frac{2+x}{4}, & -2 < x \leq 0, \\ \frac{2-x}{4}, & 0 < x \leq 2, \\ 0, & x > 2\ . \end{cases}$$

For $f * f * f$ we find

$$(f * f * f)(x) = \int\limits_{x-1}^{x+1} \frac{1}{2}\, (f * f)(z)\, \mathrm{d}z$$

$$= \begin{cases} 0, & x \leq -3, \\ \int_{-2}^{x+1} \frac{2+z}{8}\, \mathrm{d}z, & -3 < x \leq -1, \\ \int_{x-1}^{0} \frac{2+z}{8}\, \mathrm{d}z + \int\limits_{0}^{x+1} \frac{2-z}{8}\, \mathrm{d}z, & -1 < x \leq 1, \\ \int_{x-1}^{2} \frac{2-z}{8}\, \mathrm{d}z, & 1 < x \leq 3, \\ 0, & x > 3\ ; \end{cases}$$

$$(f * f * f)(x) = \begin{cases} 0, & x \leq -3, \\ \frac{x^2+6x+9}{16}, & -3 < x \leq -1, \\ \frac{6-2x^2}{16}, & -1 < x \leq 1, \\ \frac{x^2-6x+9}{16}, & 1 < x \leq 3, \\ 0, & x > 3\ . \end{cases}$$

Finally, for $f * f * f * f$ one gets

$$(f * f * f * f)(x) = \int\limits_{x-1}^{x+1} \frac{1}{2}\,(f * f * f)(z)\,\mathrm{d}z$$

$$= \begin{cases} 0, & x \le -4, \\ \int_{-3}^{x+1} \frac{z^2+6z+9}{32}\,\mathrm{d}z, & -4 < x \le -2, \\ \int_{x-1}^{-1} \frac{z^2+6z+9}{32}\,\mathrm{d}z + \int_{-1}^{x+1} \frac{6-2z^2}{32}\,\mathrm{d}z, & -2 < x \le 0, \\ \int_{x-1}^{1} \frac{6-2z^2}{32}\,\mathrm{d}z + \int_{1}^{x+1} \frac{z^2-6z+9}{32}\,\mathrm{d}z, & 0 < x \le 2, \\ \int_{x-1}^{3} \frac{z^2-6z+9}{32}\,\mathrm{d}z, & 2 < x \le 4, \\ 0, & x > 4\ ; \end{cases}$$

$$(f * f * f * f)(x) = \begin{cases} 0, & x \le -4, \\ \frac{x^3+12x^2+48x+64}{96}, & -4 < x \le -2, \\ \frac{-3x^3-12x^2+32}{96}, & -2 < x \le 0, \\ \frac{3x^3-12x^2+32}{96}, & 0 < x \le 2, \\ \frac{-x^3+12x^2-48x+64}{96}, & 2 < x \le 4, \\ 0, & x > 4\ . \end{cases}$$

Problem 3 (Properties of the Discrete Convolution)

(a) (Linearity) Show $(\alpha \cdot f + \beta \cdot g) * w = \alpha \cdot (f * w) + \beta \cdot (g * w)$, for all $\alpha, \beta \in \mathbb{R}$.

Solution:

$$\begin{aligned} (\alpha \cdot f + \beta \cdot g) * w &\overset{(1)}{=} \sum_{k=-\infty}^{\infty} (\alpha \cdot f_{i-k} + \beta \cdot g_{i-k}) \cdot w_k \\ &\overset{(2,3)}{=} \alpha \cdot \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k + \beta \cdot \sum_{k=-\infty}^{\infty} g_{i-k} \cdot w_k \\ &\overset{(1)}{=} \alpha \cdot (f * w) + \beta \cdot (g * w) \end{aligned}$$

(1) by definition of discrete convolution in 1-D.
(2) with the distributive property of multiplication over scalar addition.
(3) with the factorisation of the scalars $\alpha, \beta \in \mathbb{R}$.

(b) (Commutativity) Show $f * w = w * f$.

Solution:

$$\begin{aligned} f * w &\overset{(1)}{=} \sum_{k=-\infty}^{\infty} f_{i-k} \cdot w_k \\ &\overset{(4)}{=} \sum_{m=\infty}^{-\infty} f_m \cdot w_{i-m} \\ &\overset{(5,6)}{=} \sum_{m=-\infty}^{\infty} w_{i-m} \cdot f_m \\ &\overset{(1)}{=} w * f \end{aligned}$$

(4) with the change of variables $m = i - k$.
(5) with the commutative property of scalar addition (to invert the order of summation).
(6) with the commutative property of scalar multiplication.

(c) (Identity) For which signal e does $f * e = f$ hold?

Solution:

$$(f * e)_i \overset{(1)}{=} \sum_{k=-\infty}^{\infty} f_{i-k} \cdot e_k \overset{!}{=} f_i \,.$$

The last equality holds if and only if $e = 1$ at the point $k = 0$, and $e = 0$ elsewhere. Therefore, $f * e = f$ holds when the signal e corresponds to the delta function

$$e(x) = \delta(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{else} \end{cases}$$

Problem 4 (Continuous Fourier Transform)

(a) There are two ways to determine the Fourier transform of the hat function

$$f = \begin{cases} \frac{2-|x|}{4}, & -2 \le x \le 2 \\ 0, & \text{else} \end{cases} .$$

The first one is the direct computation via

$$
\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} f(x) \cdot exp^{(-i2\pi ux)}\ dx \\
&= \int_{-2}^{0} \underbrace{\frac{2+x}{4}}_{u} \cdot \underbrace{exp^{(-i2\pi ux)}}_{dv}\ dx + \int_{0}^{2} \underbrace{\frac{2-x}{4}}_{u} \cdot \underbrace{exp^{(-i2\pi ux)}}_{dv}\ dx \\
&= \big[\frac{2+x}{4} \cdot \frac{-1}{i2\pi u} exp^{(-i2\pi ux)}\big]_{-2}^{0} - \int_{-2}^{0} \frac{1}{4} \cdot \frac{-1}{i2\pi u} \cdot exp^{(-i2\pi ux)}\ dx \\
&+ \big[\frac{2-x}{4} \cdot \frac{-1}{i2\pi u} exp^{(-i2\pi ux)}\big]_{0}^{2} - \int_{0}^{2} \frac{-1}{4} \cdot \frac{-1}{i2\pi u} \cdot exp^{(-i2\pi ux)}\ dx \\
&= -\frac{-1}{i8\pi u}\big[\frac{-1}{i2\pi u} exp^{(-i2\pi ux)}\big]_{-2}^{0} + \frac{-1}{i8\pi u}\big[\frac{-1}{i2\pi u} exp^{(-i2\pi ux)}\big]_{0}^{2} \\
&= \frac{1}{16\pi^2 u^2}\big(1 - e^{i4\pi u}\big) - \frac{1}{16\pi^2 u^2}\big(e^{-i4\pi u} - 1\big) \\
&= \frac{1}{16\pi^2 u^2}\big(-e^{i4\pi u} - e^{-i4\pi u} + 2\big) \\
&= \frac{1}{16\pi^2 u^2}\big(-(e^{i2\pi u})^2 - (e^{-i2\pi u})^2 + 2\big) \\
&= \frac{1}{16\pi^2 u^2}\big(-(cos(2\pi u) + i \cdot sin(2\pi u))^2 - (cos(2\pi u) - i \cdot sin(2\pi u))^2 + 2\big) \\
&= \frac{1}{16\pi^2 u^2}\big(-(cos^2(2\pi u) + 2i \cdot cos(2\pi u) \cdot sin(2\pi u) - sin^2(2\pi u)) \\
&\quad - (cos^2(2\pi u) - 2i \cdot cos(2\pi u) \cdot sin(2\pi u) - sin^2(2\pi u)) + 2\big) \\
&= \frac{1}{16\pi^2 u^2}\big(-2cos^2(2\pi u) + 2sin^2(2\pi u) + 2\big) \\
&= \frac{1}{8\pi^2 u^2}\big(-(1 - sin^2(2\pi u)) + sin^2(2\pi u) + 1\big) \\
&= \frac{1}{8\pi^2 u^2} \cdot 2sin^2(2\pi u) \\
&= \left(\frac{sin(2\pi u)}{2\pi u}\right)^2 \\
&= \mathrm{sinc}^2(2\pi u)\ .
\end{aligned}
$$

Alternatively, one can use the fact that the hat function f can be written as

$$f = g * g$$

where g is a box function of type

$$g \quad = \quad \begin{cases} \frac{1}{2}, & -1 \leq x \leq 1 \\ 0, & \text{else} \end{cases} \quad .$$

This has been shown in Problem 2(a). The convolution theorem tells us that the Fourier transform $\widehat{f}(u)$ can then be computed via

$$\widehat{f}(u) = \widehat{g}(u) \cdot \widehat{g}(u) \,.$$

It is thus sufficient to determine the Fourier transform of the box-function g. Analogously to Lecture 03, Slide 8 it is given by

$$\begin{aligned}
\widehat{g}(u) &= \int_{-\infty}^{\infty} g(x) \cdot \exp(-i2\pi ux)\; dx \\
&= \int_{-1}^{1} \frac{1}{2} \cdot \exp(-i2\pi ux)\; dx \\
&= \frac{1}{2} \cdot \frac{1}{-i2\pi u} \cdot [\exp(-i2\pi ux)]_{-1}^{1} \\
&= \frac{-1}{i4\pi u} \cdot (\exp(-i2\pi u) - \exp(i2\pi u)) \\
&= \frac{-1}{i4\pi u} \cdot (\cos(-2\pi u) + i \cdot \sin(-2\pi u) - \cos(2\pi u) - i \cdot \sin(2\pi u)) \\
&= \frac{-1}{i4\pi u} \cdot (-2i \cdot \sin(2\pi u)) \\
&= \frac{1}{2\pi u} \cdot \sin(2\pi u) = \operatorname{sinc}(2\pi u)
\end{aligned}$$

The fourier transform of f then reads:

$$\begin{aligned}
\widehat{f}(u) &= \widehat{g}(u) \cdot \widehat{g}(u) \\
&= \frac{1}{2\pi u} \cdot \sin(2\pi u) \cdot \frac{1}{2\pi u} \cdot \sin(2\pi u) \\
&= \operatorname{sinc}^2(2\pi u) \,.
\end{aligned}$$

Not surprisingly, this confirms our result from the direct computation.

(b) Finally, we are in the position to compute the Fourier spectrum of the hat function. Since the Fourier transform $\widehat{f}(u)$ is real-valued in the case of the hat function, it is indentical to the Fourier spectrum $|\widehat{f}(u)|$:

$$\begin{aligned}
\left|\widehat{f}(u)\right| &= \sqrt{\operatorname{Re}^2(\widehat{f}(u)) + \operatorname{Im}^2(\widehat{f}(u))} \\
&= \sqrt{\left(\operatorname{sinc}^2(2\pi u)\right)^2} \\
&= \operatorname{sinc}^2(2\pi u) \,.
\end{aligned}$$