Prof. Dr. Joachim Weickert **Winter Term 2007/08**

Image Processing and Computer Vision

Solutions to Self-Test Problems

Problem 1 (Fourier and wavelet transform)

(a) Direct application of the Fourier transform equation gives

$$
\begin{aligned}
\hat{f}_0 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + 1f_1 + 1f_2 + 1f_3 + 1f_4 + 1f_5 + 1f_6 + 1f_7\Big)\\
\hat{f}_1 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + \tfrac{1-\mathrm{i}}{\sqrt{2}}f_1 + (-\mathrm{i})f_2 + \tfrac{-1-\mathrm{i}}{\sqrt{2}}f_3 + (-1)f_4 + \tfrac{-1+\mathrm{i}}{\sqrt{2}}f_5 + \mathrm{i}f_6 + \tfrac{1+\mathrm{i}}{\sqrt{2}}f_7\Big)\\
\hat{f}_2 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + (-\mathrm{i})f_1 + (-1)f_2 + \mathrm{i}f_3 + 1f_4 + (-\mathrm{i})f_5 + (-1)f_6 + \mathrm{i}f_7\Big)\\
\hat{f}_3 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + \tfrac{-1-\mathrm{i}}{\sqrt{2}}f_1 + \mathrm{i}f_2 + \tfrac{1-\mathrm{i}}{\sqrt{2}}f_3 + (-1)f_4 + \tfrac{1+\mathrm{i}}{\sqrt{2}}f_5 + (-\mathrm{i})f_6 + \tfrac{-1+\mathrm{i}}{\sqrt{2}}f_7\Big)\\
\hat{f}_4 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + (-1)f_1 + 1f_2 + (-1)f_3 + 1f_4 + (-1)f_5 + 1f_6 + (-1)f_7\Big)\\
\hat{f}_5 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + \tfrac{-1+\mathrm{i}}{\sqrt{2}}f_1 + (-\mathrm{i})f_2 + \tfrac{1+\mathrm{i}}{\sqrt{2}}f_3 + (-1)f_4 + \tfrac{1-\mathrm{i}}{\sqrt{2}}f_5 + \mathrm{i}f_6 + \tfrac{-1-\mathrm{i}}{\sqrt{2}}f_7\Big)\\
\hat{f}_6 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + \mathrm{i}f_1 + (-1)f_2 + (-\mathrm{i})f_3 + 1f_4 + \mathrm{i}f_5 + (-1)f_6 + (-\mathrm{i})f_7\Big)\\
\hat{f}_7 &= \tfrac{1}{\sqrt{8}}\Big(1f_0 + \tfrac{1+\mathrm{i}}{\sqrt{2}}f_1 + \mathrm{i}f_2 + \tfrac{-1+\mathrm{i}}{\sqrt{2}}f_3 + (-1)f_4 + \tfrac{-1-\mathrm{i}}{\sqrt{2}}f_5 + (-\mathrm{i})f_6 + \tfrac{1-\mathrm{i}}{\sqrt{2}}f_7\Big)
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\hat{f}_0 &= \tfrac{17}{2}\sqrt{2} &&\approx 12.02081528\\
\hat{f}_1 &= \big(-\tfrac{5}{2}+\tfrac{3}{2}\sqrt{2}\big)+\tfrac{3}{2}\sqrt{2}\mathrm{i} &&\approx -0.37867966+2.12132034\mathrm{i}\\
\hat{f}_2 &= -\tfrac{1}{2}\sqrt{2}+\tfrac{1}{2}\sqrt{2}\mathrm{i} &&\approx -0.70710678+0.70710678\mathrm{i}\\
\hat{f}_3 &= \big(\tfrac{5}{2}+\tfrac{3}{2}\sqrt{2}\big)-\tfrac{3}{2}\sqrt{2}\mathrm{i} &&\approx 4.62132034-2.12132034\mathrm{i}\\
\hat{f}_4 &= \tfrac{1}{2}\sqrt{2} &&\approx 0.70710678\\
\hat{f}_5 &= \big(\tfrac{5}{2}+\tfrac{3}{2}\sqrt{2}\big)+\tfrac{3}{2}\sqrt{2}\mathrm{i} &&\approx 4.62132034+2.12132034\mathrm{i}\\
\hat{f}_6 &= -\tfrac{1}{2}\sqrt{2}-\tfrac{1}{2}\sqrt{2}\mathrm{i} &&\approx -0.70710678-0.70710678\mathrm{i}\\
\hat{f}_7 &= \big(-\tfrac{5}{2}+\tfrac{3}{2}\sqrt{2}\big)-\tfrac{3}{2}\sqrt{2}\mathrm{i} &&\approx -0.37867966-2.12132034\mathrm{i}
\end{aligned}
$$

The highest frequency is represented by the coefficient $\hat{f}_4$. Elimination and back-transformation (analogous to forward transform, with complex conjugate coefficients) yields

$$\tilde{f} = \left(\frac{27}{4},\ \frac{5}{4},\ \frac{7}{4},\ \frac{29}{4},\ \frac{3}{4},\ \frac{25}{4},\ \frac{31}{4},\ \frac{9}{4}\right)$$
$$\approx (6.75,\ 1.25,\ 1.75,\ 7.25,\ 0.75,\ 6.25,\ 7.75,\ 2.25)\ .$$

(b) Starting from

$$c_{0,0} = 7,\quad c_{0,1} = 1,\quad c_{0,2} = 2,\quad c_{0,3} = 7,\quad c_{0,4} = 1,\quad c_{0,5} = 6,\quad c_{0,6} = 8,\quad c_{0,7} = 2$$

we use the formulae from Lecture 6 to compute

$$c_{1,0} = 4\sqrt{2},\qquad c_{1,1} = \frac{9}{2}\sqrt{2},\qquad c_{1,2} = \frac{7}{2}\sqrt{2},\qquad c_{1,3} = 5\sqrt{2},$$
$$d_{1,0} = 3\sqrt{2},\qquad d_{1,1} = -\frac{5}{2}\sqrt{2},\qquad d_{1,2} = -\frac{5}{2}\sqrt{2},\qquad d_{1,3} = 3\sqrt{2},$$

$$c_{2,0} = \frac{17}{2},\qquad c_{2,1} = \frac{17}{2},$$
$$d_{2,0} = -\frac{1}{2},\qquad d_{2,1} = -\frac{3}{2},$$

$$c_{3,0} = \frac{17}{2}\sqrt{2},$$
$$d_{3,0} = 0.$$

The wavelet representation consists of the scaling coefficient $c_{3,0}$ and the seven wavelet coefficients, i.e.

$$(c_{3,0},\ d_{3,0},\ d_{2,0},\ d_{2,1},\ d_{1,0},\ d_{1,1},\ d_{1,2},\ d_{1,3})$$
$$= \left(\frac{17}{2}\sqrt{2},\ 0,\ -\frac{1}{2},\ -\frac{3}{2},\ 3\sqrt{2},\ -\frac{5}{2}\sqrt{2},\ -\frac{5}{2}\sqrt{2},\ 3\sqrt{2}\right)$$
$$\approx (12.02081528,\ 0,\ -0.5,\ -1.5,\ 4.24264069,\ -3.53553391,\ -3.53553391,\ 4.24264069)$$

The wavelet coefficient with smallest absolute value is $d_{3,0}$ which is already zero. Therefore nothing changes by eliminating it. The back-transform therefore reproduces exactly the above values $c_{j,k}$ and finally restores the original signal.

Problem 2 (Histogram equalisation)

Note that the algorithm from the lecture uses grey value indices starting from 1 while the grey values of the given table start from 0. Therefore grey value *index* k always represents the grey value $k-1$.

We calculate then the cumulated frequencies $\sum p_i$ of given pixels up to grey value index k and the desired cumulated frequencies $\sum q_j$ up to grey value index r. The ordering between the two sequences of sums implies the grey value maps.

Index k	$\sum_{i=1}^{k} p_i$	Index r	$\sum_{j=1}^{r} q_j$	Index map	Grey value map
		1	0.125		
1	0.17	2	0.25	$1 \to 2$	$0 \to 1$
2	0.36	3	0.375	$2 \to 3$	$1 \to 2$
		4	0.5		
		5	0.625		
3	0.63	6	0.75	$3 \to 6$	$2 \to 5$
4	0.84	7	0.875	$4 \to 7$	$3 \to 6$
5	0.90				
6	0.94				
7	0.97				
8	1.00	8	1.0	$5, 6, 7, 8 \to 8$	$4, 5, 6, 7 \to 7$

The resulting histogram is

Grey value (new)	relative frequency
0	0.00
1	0.17
2	0.19
3	0.00
4	0.00
5	0.27
6	0.21
7	0.16

Problem 3 (Motion analysis)

(a) Since we speak of 1-D images, the spatial gradient is exactly the spatial derivative f_x.

The assumption that corresponding object points in subsequent frames have equal gradients is therefore expressed by the equation

$$f_x(x+u, z+1) = f_x(x, z)$$

where u denotes the optic flow in 1-D, and z denotes time. Linearisation gives in full analogy with the lecture

$$\begin{aligned} 0 &= f_x(x+u, z+1) - f_x(x, z) \\ &\approx f_{xx}(x, z)\, u + f_{xz}(x, z) \;. \end{aligned}$$

The desired OFC is therefore

$$f_{xx}(x, z)\, u + f_{xz}(x, z) = 0 \;.$$

(b) The data term for the Horn–Schunck energy functional is obtained as usual by squaring the OFC expression. The smoothness term penalises the squared gradient norm – remembering that gradients in 1-D are just spatial derivatives we arrive at

$$E(u) = \int\limits_{\Omega} \left((f_{xx}(x, z)\, u(x, z) + f_{xz}(x, z))^2 + \alpha\, (u_x(x, z))^2 \right) \, \mathrm{d}x \;.$$

(c) The integrand in the energy functional is

$$F(x, z, u, u_x) = \left((f_{xx}(x, z)\, u(x, z) + f_{xz}(x, z))^2 + \alpha\, (u_x(x, z))^2 \right) \;.$$

Inserting this in the usual Euler–Lagrange formalism we find successively

$$\begin{aligned} \frac{\partial F}{\partial u} &= 2 f_{xx} (f_{xx}\, u + f_{xz}) \\ \frac{\partial F}{\partial u_x} &= 2\alpha u_x \\ \frac{\mathrm{d}}{\mathrm{d}x} \frac{\partial F}{\partial u_x} &= 2\alpha u_{xx} \end{aligned}$$

such that the Euler–Lagrange equation reads

$$2 f_{xx} (f_{xx}\, u + f_{xz}) - 2\alpha u_{xx} = 0$$

or, after division by -2α,

$$u_{xx} - \frac{1}{\alpha} f_{xx} (f_{xx} u + f_{xz}) = 0 \;.$$

Problem 4 (Morphological operations)

The correspondence is

$$\begin{array}{ccc} \mathbf{(A)} \mathrel{\hat{=}} \mathbf{(1)}, & \mathbf{(B)} \mathrel{\hat{=}} \mathbf{(4)}, & \mathbf{(C)} \mathrel{\hat{=}} \mathbf{(6)} \\ \mathbf{(D)} \mathrel{\hat{=}} \mathbf{(5)}, & \mathbf{(E)} \mathrel{\hat{=}} \mathbf{(3)}, & \mathbf{(F)} \mathrel{\hat{=}} \mathbf{(2)}. \end{array}$$

Problem 5 (Derivative filter)

(a) As usual we abbreviate the derivatives of f at pixel i by $f_i', f_i'', \ldots$ Substituting the Taylor expansions

$$\begin{aligned}
f_{i-2} &= f_i - 2hf_i' + 2h^2 f_i'' - \tfrac{4}{3}h^3 f_i''' + \tfrac{2}{3}h^4 f_i'''' + \mathcal{O}(h^5) \\
f_{i-1} &= f_i - hf_i' + \tfrac{1}{2}h^2 f_i'' - \tfrac{1}{6}h^3 f_i''' + \tfrac{1}{24}h^4 f_i'''' + \mathcal{O}(h^5) \\
f_i &= f_i \\
f_{i+1} &= f_i + hf_i' + \tfrac{1}{2}h^2 f_i'' + \tfrac{1}{6}h^3 f_i''' + \tfrac{1}{24}h^4 f_i'''' + \mathcal{O}(h^5) \\
f_{i-2} &= f_i + 2hf_i' + 2h^2 f_i'' + \tfrac{4}{3}h^3 f_i''' + \tfrac{2}{3}h^4 f_i'''' + \mathcal{O}(h^5)
\end{aligned}$$

into an ansatz

$$f_i'' = \alpha_{-2} f_{i-2} + \alpha_{-1} f_{i-1} + \alpha_0 f_i + \alpha_1 f_{i+1} + \alpha_2 f_{i+2}$$

and neglecting the higher order error terms, we obtain

$$\begin{aligned}
f_i'' = {} & (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2) f_i \\
& + (-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2) h f_i' \\
& + (2\alpha_{-2} + \tfrac{1}{2}\alpha_{-1} + \tfrac{1}{2}\alpha_1 + 2\alpha_2) h^2 f_i'' \\
& + (-\tfrac{4}{3}\alpha_{-2} - \tfrac{1}{6}\alpha_{-1} + \tfrac{1}{6}\alpha_1 + \tfrac{4}{3}\alpha_2) h^3 f_i''' \\
& + (\tfrac{2}{3}\alpha_{-2} + \tfrac{1}{24}\alpha_{-1} + \tfrac{1}{24}\alpha_1 + \tfrac{2}{3}\alpha_2) h^4 f_i''''
\end{aligned}$$

and by comparing coefficients the desired system of equations, in matrix notation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 2h^2 & \frac{1}{2}h^2 & 0 & \frac{1}{2}h^2 & 2h^2 \\ -\frac{4}{3}h^3 & -\frac{1}{6}h^3 & 0 & \frac{1}{6}h^3 & \frac{4}{3}h^3 \\ \frac{2}{3}h^4 & \frac{1}{24}h^4 & 0 & \frac{1}{24}h^4 & \frac{2}{3}h^4 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

or equivalently

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ h^{-2} \\ 0 \\ 0 \end{pmatrix}$$

(b) We solve the system of equations obtained in (a) and find

$$
\begin{aligned}
\alpha_{-2} = \alpha_2 &= -\frac{1}{12h^2}, \\
\alpha_{-1} = \alpha_1 &= \frac{4}{3h^2}, \\
\alpha_0 &= -\frac{5}{2h^2} \,.
\end{aligned}
$$

(c) Applying the resulting approximation

$$
f_i'' \approx \frac{-f_{i-2} + 16f_{i-1} - 30f_i + 16f_{i+1} - f_{i+2}}{12h^2}
$$

to the given signal f leads to

$$
f'' \approx \left(\ldots, 0, 0, 0, \frac{-2}{h^2}, \frac{30}{h^2}, \frac{-30}{h^2}, \frac{2}{h^2}, 0, 0, 0, \ldots\right) \,.
$$

Problem 6 (Moments)

(a) The four moments $m_{0,0}$, $m_{1,0}$, $m_{0,1}$ and $m_{1,1}$ are given by

$$\begin{aligned}
m_{0,0} &= 1\cdot 1\cdot 2+1\cdot 1\cdot 1+1\cdot 1\cdot 3+1\cdot 1\cdot 5+1\cdot 1\cdot 4+1\cdot 1\cdot 4=19,\\
m_{1,0} &= 1\cdot 1\cdot 2+2\cdot 1\cdot 1+3\cdot 1\cdot 3+1\cdot 1\cdot 5+2\cdot 1\cdot 4+3\cdot 1\cdot 4=38,\\
m_{0,1} &= 1\cdot 1\cdot 2+1\cdot 1\cdot 1+1\cdot 1\cdot 3+1\cdot 2\cdot 5+1\cdot 2\cdot 4+1\cdot 2\cdot 4=32,\\
m_{1,1} &= 1\cdot 1\cdot 2+2\cdot 1\cdot 1+3\cdot 1\cdot 3+1\cdot 2\cdot 5+2\cdot 2\cdot 4+3\cdot 2\cdot 4=63.
\end{aligned}$$

(b) The four central moments $\mu_{0,0}$, $\mu_{1,0}$, $\mu_{0,1}$ and $\mu_{1,1}$ are given by

$$\begin{aligned}
\mu_{0,0} &= m_{0,0}=19,\\
\mu_{1,0} &= 0,\\
\mu_{0,1} &= 0,\\
\mu_{1,1} &= m_{1,1}-\bar{j}m_{1,0}=m_{1,1}-\frac{m_{0,1}}{m_{0,0}}\,m_{1,0}=63-\frac{32\cdot 38}{19}=-1.
\end{aligned}$$

(c) The given relation can be proven in the following way:

$$\begin{aligned}
\mu_{1,1} &= \sum_{i=1}^{N}\sum_{j=1}^{M}(i-\bar{i})\,(j-\bar{j})\,f_{ij}\\
&= \sum_{i=1}^{N}\sum_{j=1}^{M}(ij-\bar{i}j-\bar{j}i+\bar{i}\bar{j})\,f_{ij}\\
&= \sum_{i=1}^{N}\sum_{j=1}^{M}ijf_{ij}-\bar{i}\sum_{i=1}^{N}\sum_{j=1}^{M}jf_{ij}-\bar{j}\sum_{i=1}^{N}\sum_{j=1}^{M}if_{ij}+\bar{i}\bar{j}\sum_{i=1}^{N}\sum_{j=1}^{M}f_{ij}\\
&= m_{1,1}-\bar{i}\,m_{0,1}-\bar{j}\,m_{1,0}+\bar{i}\,\bar{j}\,m_{0,0}\\
&= m_{1,1}-\frac{m_{1,0}}{m_{0,0}}\,m_{0,1}-\frac{m_{0,1}}{m_{0,0}}\,m_{1,0}+\frac{m_{1,0}}{m_{0,0}}\,\frac{m_{0,1}}{m_{0,0}}\,m_{0,0}\\
&= m_{1,1}-\frac{m_{1,0}}{m_{0,0}}\,m_{0,1}\\
&= m_{1,1}-\bar{i}\,m_{0,1}.
\end{aligned}$$