

Image Processing and Computer Vision

Solutions to Self-Test Problems

Problem 1 (Fourier and wavelet transform)

(a)

Direct application of the Fourier transform equation gives

$$\begin{aligned}\hat{f}_0 &= \frac{1}{\sqrt{8}} \left(1f_0 + 1f_1 + 1f_2 + 1f_3 + 1f_4 + 1f_5 + 1f_6 + 1f_7 \right) \\ \hat{f}_1 &= \frac{1}{\sqrt{8}} \left(1f_0 + \frac{1-i}{\sqrt{2}}f_1 + (-i)f_2 + \frac{-1-i}{\sqrt{2}}f_3 + (-1)f_4 + \frac{-1+i}{\sqrt{2}}f_5 + if_6 + \frac{1+i}{\sqrt{2}}f_7 \right) \\ \hat{f}_2 &= \frac{1}{\sqrt{8}} \left(1f_0 + (-i)f_1 + (-1)f_2 + if_3 + 1f_4 + (-i)f_5 + (-1)f_6 + if_7 \right) \\ \hat{f}_3 &= \frac{1}{\sqrt{8}} \left(1f_0 + \frac{-1-i}{\sqrt{2}}f_1 + if_2 + \frac{1-i}{\sqrt{2}}f_3 + (-1)f_4 + \frac{1+i}{\sqrt{2}}f_5 + (-i)f_6 + \frac{-1+i}{\sqrt{2}}f_7 \right) \\ \hat{f}_4 &= \frac{1}{\sqrt{8}} \left(1f_0 + (-1)f_1 + 1f_2 + (-1)f_3 + 1f_4 + (-1)f_5 + 1f_6 + (-1)f_7 \right) \\ \hat{f}_5 &= \frac{1}{\sqrt{8}} \left(1f_0 + \frac{-1+i}{\sqrt{2}}f_1 + (-i)f_2 + \frac{1+i}{\sqrt{2}}f_3 + (-1)f_4 + \frac{1-i}{\sqrt{2}}f_5 + if_6 + \frac{-1-i}{\sqrt{2}}f_7 \right) \\ \hat{f}_6 &= \frac{1}{\sqrt{8}} \left(1f_0 + if_1 + (-1)f_2 + (-i)f_3 + 1f_4 + if_5 + (-1)f_6 + (-i)f_7 \right) \\ \hat{f}_7 &= \frac{1}{\sqrt{8}} \left(1f_0 + \frac{1+i}{\sqrt{2}}f_1 + if_2 + \frac{-1+i}{\sqrt{2}}f_3 + (-1)f_4 + \frac{-1-i}{\sqrt{2}}f_5 + (-i)f_6 + \frac{1-i}{\sqrt{2}}f_7 \right)\end{aligned}$$

and therefore

$$\begin{aligned}\hat{f}_0 &= \frac{17}{2}\sqrt{2} \approx 12.02081528 \\ \hat{f}_1 &= \left(-\frac{5}{2} + \frac{3}{2}\sqrt{2}\right) + \frac{3}{2}\sqrt{2}i \approx -0.37867966 + 2.12132034i \\ \hat{f}_2 &= -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i \approx -0.70710678 + 0.70710678i \\ \hat{f}_3 &= \left(\frac{5}{2} + \frac{3}{2}\sqrt{2}\right) - \frac{3}{2}\sqrt{2}i \approx 4.62132034 - 2.12132034i \\ \hat{f}_4 &= \frac{1}{2}\sqrt{2} \approx 0.70710678 \\ \hat{f}_5 &= \left(\frac{5}{2} + \frac{3}{2}\sqrt{2}\right) + \frac{3}{2}\sqrt{2}i \approx 4.62132034 + 2.12132034i \\ \hat{f}_6 &= -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i \approx -0.70710678 - 0.70710678i \\ \hat{f}_7 &= \left(-\frac{5}{2} + \frac{3}{2}\sqrt{2}\right) - \frac{3}{2}\sqrt{2}i \approx -0.37867966 - 2.12132034i\end{aligned}$$

The highest frequency is represented by the coefficient \hat{f}_4 . Elimination and back-transformation (analogous to forward transform, with complex conjugate coefficients) yields

$$\begin{aligned}\tilde{f} &= \left(\frac{27}{4}, \frac{5}{4}, \frac{7}{4}, \frac{29}{4}, \frac{3}{4}, \frac{25}{4}, \frac{31}{4}, \frac{9}{4} \right) \\ &\approx (6.75, 1.25, 1.75, 7.25, 0.75, 6.25, 7.75, 2.25).\end{aligned}$$

(b)

Starting from

$$c_{0,0} = 7, \quad c_{0,1} = 1, \quad c_{0,2} = 2, \quad c_{0,3} = 7, \quad c_{0,4} = 1, \quad c_{0,5} = 6, \quad c_{0,6} = 8, \quad c_{0,7} = 2$$

we use the formulae from Lecture 6 to compute

$$\begin{aligned}c_{1,0} &= 4\sqrt{2}, & c_{1,1} &= \frac{9}{2}\sqrt{2}, & c_{1,2} &= \frac{7}{2}\sqrt{2}, & c_{1,3} &= 5\sqrt{2}, \\ d_{1,0} &= 3\sqrt{2}, & d_{1,1} &= -\frac{5}{2}\sqrt{2}, & d_{1,2} &= -\frac{5}{2}\sqrt{2}, & d_{1,3} &= 3\sqrt{2}, \\ c_{2,0} &= \frac{17}{2}, & c_{2,1} &= \frac{17}{2}, \\ d_{2,0} &= -\frac{1}{2}, & d_{2,1} &= -\frac{3}{2}, \\ c_{3,0} &= \frac{17}{2}\sqrt{2}, \\ d_{3,0} &= 0.\end{aligned}$$

The wavelet representation consists of the scaling coefficient $c_{3,0}$ and the seven wavelet coefficients, i.e.

$$\begin{aligned}&(c_{3,0}, d_{3,0}, d_{2,0}, d_{2,1}, d_{1,0}, d_{1,1}, d_{1,2}, d_{1,3}) \\ &= \left(\frac{17}{2}\sqrt{2}, 0, -\frac{1}{2}, -\frac{3}{2}, 3\sqrt{2}, -\frac{5}{2}\sqrt{2}, -\frac{5}{2}\sqrt{2}, 3\sqrt{2} \right) \\ &\approx (12.02081528, 0, -0.5, -1.5, \\ &\quad 4.24264069, -3.53553391, -3.53553391, 4.24264069)\end{aligned}$$

The wavelet coefficient with smallest absolute value is $d_{3,0}$ which is already zero. Therefore nothing changes by eliminating it. The back-transform therefore reproduces exactly the above values $c_{j,k}$ and finally restores the original signal.

Problem 2 (Histogram equalisation)

Note that the algorithm from the lecture uses greyvalue indices starting from 1 while the greyvalues of the given table start from 0. Therefore greyvalue *index* k always represents the greyvalue $k - 1$.

We calculate then the cumulated frequencies $\sum p_i$ of given pixels up to greyvalue index k and the desired cumulated frequencies $\sum q_j$ up to greyvalue index r . The ordering between the two sequences of sums implies the greyvalue maps.

Index k	$\sum_{i=1}^k p_i$	Index r	$\sum_{j=1}^r q_j$	Index map	Greyvalue map
		1	0.125		
1	0.17	2	0.25	$1 \rightarrow 2$	$0 \rightarrow 1$
2	0.36	3	0.375	$2 \rightarrow 3$	$1 \rightarrow 2$
		4	0.5		
		5	0.625		
3	0.63	6	0.75	$3 \rightarrow 6$	$2 \rightarrow 5$
4	0.84	7	0.875	$4 \rightarrow 7$	$3 \rightarrow 6$
5	0.90				
6	0.94				
7	0.97				
8	1.00	8	1.0	$5, 6, 7, 8 \rightarrow 8$	$4, 5, 6, 7 \rightarrow 7$

The resulting histogram is

Greyvalue (new)	relative frequency
0	0.00
1	0.17
2	0.19
3	0.00
4	0.00
5	0.27
6	0.21
7	0.16

Problem 3 (Motion analysis)

(a)

Since we speak of 1-D images, the spatial gradient is exactly the spatial derivative f_x .

The assumption that corresponding object points in subsequent frames have equal gradients is therefore expressed by the equation

$$f_x(x + u, z + 1) = f_x(x, z)$$

where u denotes the optic flow in 1-D, and z denotes time. Linearisation gives in full analogy with the lecture

$$\begin{aligned} 0 &= f_x(x + u, z + 1) - f_x(x, z) \\ &\approx f_{xx}(x, z) u + f_{xz}(x, z) . \end{aligned}$$

The desired OFC is therefore

$$f_{xx}(x, z) u + f_{xz}(x, z) = 0 .$$

(b)

The data term for the Horn–Schunck energy functional is obtained as usual by squaring the OFC expression. The smoothness term penalises the squared gradient norm – remembering that gradients in 1-D are just spatial derivatives we arrive at

$$E(u) = \int_{\Omega} ((f_{xx}(x, z) u(x, z) + f_{xz}(x, z))^2 + \alpha (u_x(x, z))^2) dx .$$

(c)

The integrand in the energy functional is

$$F(x, z, u, u_x) = ((f_{xx}(x, z) u(x, z) + f_{xz}(x, z))^2 + \alpha (u_x(x, z))^2) .$$

Inserting this in the usual Euler–Lagrange formalism we find successively

$$\begin{aligned} \frac{\partial F}{\partial u} &= 2f_{xx}(f_{xx} u + f_{xz}) \\ \frac{\partial F}{\partial u_x} &= 2\alpha u_x \\ \frac{d}{dx} \frac{\partial F}{\partial u_x} &= 2\alpha u_{xx} \end{aligned}$$

such that the Euler–Lagrange equation reads

$$2f_{xx}(f_{xx} u + f_{xz}) - 2\alpha u_{xx} = 0$$

or, after division by -2α ,

$$u_{xx} - \frac{1}{\alpha} f_{xx}(f_{xx} u + f_{xz}) .$$

Problem 4 (Morphological operations)

The correspondence is

$$\begin{aligned} (\mathbf{A}) &\hat{=} (\mathbf{1}), & (\mathbf{B}) &\hat{=} (\mathbf{4}), & (\mathbf{C}) &\hat{=} (\mathbf{6}) \\ (\mathbf{D}) &\hat{=} (\mathbf{5}), & (\mathbf{E}) &\hat{=} (\mathbf{3}), & (\mathbf{F}) &\hat{=} (\mathbf{2}). \end{aligned}$$

Problem 5 (Derivative filter)

(a)

As usual we abbreviate the derivatives of f at pixel i by f'_i, f''_i, \dots . Substituting the Taylor expansions

$$\begin{aligned} f_{i-2} &= f_i - 2hf'_i + 2h^2f''_i - \frac{4}{3}h^3f'''_i + \frac{2}{3}h^4f''''_i + \mathcal{O}(h^5) \\ f_{i-1} &= f_i - hf'_i + \frac{1}{2}h^2f''_i - \frac{1}{6}h^3f'''_i + \frac{1}{24}h^4f''''_i + \mathcal{O}(h^5) \\ f_i &= f_i \\ f_{i+1} &= f_i + hf'_i + \frac{1}{2}h^2f''_i + \frac{1}{6}h^3f'''_i + \frac{1}{24}h^4f''''_i + \mathcal{O}(h^5) \\ f_{i+2} &= f_i + 2hf'_i + 2h^2f''_i + \frac{4}{3}h^3f'''_i + \frac{2}{3}h^4f''''_i + \mathcal{O}(h^5) \end{aligned}$$

into an ansatz

$$f''_i = \alpha_{-2}f_{i-2} + \alpha_{-1}f_{i-1} + \alpha_0f_i + \alpha_1f_{i+1} + \alpha_2f_{i+2}$$

and neglecting the higher order error terms, we obtain

$$\begin{aligned} f''_i &= (\alpha_{-2} + \alpha_{-1} + \alpha_0 + \alpha_1 + \alpha_2)f_i \\ &+ (-2\alpha_{-2} - \alpha_{-1} + \alpha_1 + 2\alpha_2)hf'_i \\ &+ (2\alpha_{-2} + \frac{1}{2}\alpha_{-1} + \frac{1}{2}\alpha_1 + 2\alpha_2)h^2f''_i \\ &+ (-\frac{4}{3}\alpha_{-2} - \frac{1}{6}\alpha_{-1} + \frac{1}{6}\alpha_1 + \frac{4}{3}\alpha_2)h^3f'''_i \\ &+ (\frac{2}{3}\alpha_{-2} + \frac{1}{24}\alpha_{-1} + \frac{1}{24}\alpha_1 + \frac{2}{3}\alpha_2)h^4f''''_i \end{aligned}$$

and by comparing coefficients the desired system of equations, in matrix notation:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2h & -h & 0 & h & 2h \\ 2h^2 & \frac{1}{2}h^2 & 0 & \frac{1}{2}h^2 & 2h^2 \\ -\frac{4}{3}h^3 & -\frac{1}{6}h^3 & 0 & \frac{1}{6}h^3 & \frac{4}{3}h^3 \\ \frac{2}{3}h^4 & \frac{1}{24}h^4 & 0 & \frac{1}{24}h^4 & \frac{2}{3}h^4 \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = (0 \ 0 \ 1 \ 0 \ 0)$$

or equivalently

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \alpha_{-2} \\ \alpha_{-1} \\ \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = (0 \quad 0 \quad h^{-2} \quad 0 \quad 0)$$

(b)

We solve the system of equations obtained in (a) and find

$$\begin{aligned} \alpha_{-2} = \alpha_2 &= -\frac{1}{12h^2}, \\ \alpha_{-1} = \alpha_1 &= \frac{4}{3h^2}, \\ \alpha_0 &= -\frac{5}{2h^2}. \end{aligned}$$

(c) Applying the resulting approximation

$$f_i'' \approx \frac{-f_{i-2} + 16f_{i-1} - 30f_i + 16f_{i+1} - f_{i+2}}{12h^2}$$

to the given signal f leads to

$$f'' \approx \left(\dots, 0, 0, 0, \frac{-2}{h^2}, \frac{30}{h^2}, \frac{-30}{h^2}, \frac{2}{h^2}, 0, 0, 0, \dots \right).$$

Problem 6 (Moments)**(a)**The four moments $sm_{0,0}$, $m_{1,0}$, $m_{0,1}$ and $m_{1,1}$ are given by

$$\begin{aligned}m_{0,0} &= 1 \cdot 1 \cdot 2 + 1 \cdot 1 \cdot 1 + 1 \cdot 1 \cdot 3 + 1 \cdot 1 \cdot 5 + 1 \cdot 1 \cdot 4 + 1 \cdot 1 \cdot 4 = 19, \\m_{1,0} &= 1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 3 + 1 \cdot 1 \cdot 5 + 2 \cdot 1 \cdot 4 + 3 \cdot 1 \cdot 4 = 38, \\m_{0,1} &= 1 \cdot 1 \cdot 2 + 1 \cdot 2 \cdot 1 + 1 \cdot 1 \cdot 3 + 1 \cdot 2 \cdot 5 + 1 \cdot 1 \cdot 4 + 1 \cdot 2 \cdot 4 = 29, \\m_{1,1} &= 1 \cdot 1 \cdot 2 + 2 \cdot 2 \cdot 1 + 3 \cdot 1 \cdot 3 + 1 \cdot 2 \cdot 5 + 2 \cdot 1 \cdot 4 + 3 \cdot 2 \cdot 4 = 57.\end{aligned}$$

(b)The four central moments $\mu_{0,0}$, $\mu_{1,0}$, $\mu_{0,1}$ and $\mu_{1,1}$ are given by

$$\begin{aligned}\mu_{0,0} &= m_{0,0} = 19, \\ \mu_{1,0} &= 0, \\ \mu_{0,1} &= 0, \\ \mu_{1,1} &= m_{1,1} - \bar{j}m_{1,0} = m_{1,1} - \frac{m_{0,1}}{m_{0,0}}m_{1,0} = 57 - \frac{29 \cdot 38}{19} = -1.\end{aligned}$$

(c)

The given relation can be proven in the following way:

$$\begin{aligned}\mu_{1,1} &= \sum_{i=1}^N \sum_{j=1}^M (i - \bar{i})(j - \bar{j}) f_{ij} \\ &= \sum_{i=1}^N \sum_{j=1}^M (ij - \bar{i}j - \bar{j}i + \bar{i}\bar{j}) f_{ij} \\ &= \sum_{i=1}^N \sum_{j=1}^M ij f_{ij} - \bar{i} \sum_{i=1}^N \sum_{j=1}^M j f_{ij} - \bar{j} \sum_{i=1}^N \sum_{j=1}^M i f_{ij} + \bar{i}\bar{j} \sum_{i=1}^N \sum_{j=1}^M f_{ij} \\ &= m_{1,1} - \bar{i}m_{0,1} - \bar{j}m_{1,0} + \bar{i}\bar{j}m_{0,0} \\ &= m_{1,1} - \frac{m_{1,0}}{m_{0,0}}m_{0,1} - \frac{m_{0,1}}{m_{0,0}}m_{1,0} + \frac{m_{1,0}}{m_{0,0}}\frac{m_{0,1}}{m_{0,0}}m_{0,0} \\ &= m_{1,1} - \frac{m_{1,0}}{m_{0,0}}m_{0,1} \\ &= m_{1,1} - \bar{i}m_{0,1}.\end{aligned}$$