Image Processing and Computer Vision 2005/06
Example Solutions for Theoretical Assignments 7 (T7)

Problem 1

(a) The method of Lucas and Kanade allows to compute the optic flow everywhere. - NO.

(b) The method of Horn and Schunck can be extended to 3-D. - YES.

(c) The method of Horn and Schunck requires more iterations to converge if α is larger. - YES.

(d) In homogeneous coordinates parallel lines may intersect. YES.

(e) In a non-orthoparallel (converging) camera setup, the search space increases to 2-D. - NO.

(f) In a non-orthoparallel (converging) camera setup, the disparity may be negative. YES. (due to the definition of the disparity in the script as a distance we also accept NO).

Problem 2

- A rotation by 30° around the x-axis is described by the rotation matrix

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30^\circ & -\sin 30^\circ & 0 \\ 0 & \sin 30^\circ & \cos 30^\circ & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- Similarly, a 60° rotation around the z-axis has the rotation matrix

$$R_2 = \begin{pmatrix} \cos 60^\circ & -\sin 60^\circ & 0 & 0 \\ \sin 60^\circ & \cos 60^\circ & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

- A translation by $(1, 2, -3)^\top$ in projective coordinates has the matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For successive application of these transformations, their matrices are multiplied. Remember that the multiplication must be done in reverse order.[1] We

[1] The reason is that transformation matrices act on vectors by multiplication from the left. A vector v becomes $M_1 v$ after some transformation with matrix M_1, and $M_2(M_1 v) = (M_2 M_1)v$ if M_2 is applied after M_1.

obtain therefore the projection matrix

$$
\begin{aligned}
P &= T\,R_2\,R_1 \\
&= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} & -\frac{3}{4} & \frac{\sqrt{3}}{4} & 1 \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{4} & -\frac{1}{4} & 2 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Problem 3.

(a) The inverse of the matrices A_2^{int} and A_2^{ext} are given by

$$
A_2^{\text{int}} = \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{pmatrix} \qquad
A_2^{\text{ext}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

(b) In order to compute the 2-D pixel coordinates of a 3-D point from the world coordinate system we have to deal with four coordinate systems and three transformations:

3-D world coordinates
↓ via extrinsic parameters
3-D camera coordinates
↓ via projection
2-D image coordinates
↓ via intrinsic parameters
2-D pixel coordinates

Thereby the use of homogeneous coordinates instead of Euclidean ones can simplify the computation significantly.

If we are interested in reconstructing the dept of a 3-D point, we have to perform these transformations in the reverse order. First, we have to compensate for the intrinsic parameters. To this end, we reformulate the 2-D Euclidean pixel coordinates

$$
\begin{aligned}
\mathbf{x}_1 &= (10.5, 0.5)^\top \\
\mathbf{x}_2 &= (9.5, \sqrt{2})^\top
\end{aligned}
$$

as 2-D homogeneous coordinates

$$\begin{aligned}\hat{\mathbf{x}}_1 &= (10.5, 0.5, 1)^\top \\ \hat{\mathbf{x}}_2 &= (9.5, \sqrt{2}, 1)^\top.\end{aligned}$$

Then we change to 2-D homogeneous image coordinates by applying the inverse matrix for the intrinsic parameters from taks (a). This yields:

$$\begin{aligned}(A_1^{\text{int}})^{-1}\hat{\mathbf{x}}_1 &= \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 10.5 \\ 0.5 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -9.5 \\ 1 \end{pmatrix} \\ (A_2^{\text{int}})^{-1}\hat{\mathbf{x}}_2 &= \begin{pmatrix} 1 & 0 & -10 \\ 0 & 1 & -10 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 9.5 \\ \sqrt{2} \\ 1 \end{pmatrix} = \begin{pmatrix} -0.5 \\ \sqrt{2} - 10 \\ 1 \end{pmatrix}\end{aligned}$$

The relation between this 2-D homoegeneous image coordinates with $f = 1$ and the corresponding line line 3-D Euclidean camera coordinates is very simple. The lines in 3-D Euclidean camera coordinates are given by

$$\mathbf{X_1} = \lambda_1 \begin{pmatrix} 0.5 \\ -9.5 \\ 1 \end{pmatrix} \qquad , \lambda_1 \in \mathbb{R}$$

and

$$\mathbf{X_2} = \lambda_2 \begin{pmatrix} -0.5 \\ \sqrt{2} - 10 \\ 1 \end{pmatrix} \qquad , \lambda_2 \in \mathbb{R}$$

(c) In order to intersect these rays in task (d), we have to rewrite both lines in 3-D Euclidean world coodinates. To this end we have to formulate them in homogeneous 3-D camera coordinates first. They are given by

$$\begin{aligned}\hat{\mathbf{X}}_\mathbf{1} &= (0.5\lambda_1, -9.5\lambda_1, \lambda_1, 1)^\top \qquad , \lambda_1 \in \mathbb{R} \\ \hat{\mathbf{X}}_\mathbf{2} &= (-0.5\lambda_2, -(\sqrt{2} - 10)\lambda_2, \lambda_2, 1)^\top \qquad , \lambda_2 \in \mathbb{R}.\end{aligned}$$

Then we have to compensate for the extrinsic parameters:

$$
\begin{aligned}
(A_1^{\text{ext}})^{-1}\hat{\mathbf{X}}_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5\lambda_1 \\ -9.5\lambda_1 \\ \lambda_1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.5\lambda_1 \\ -9.5\lambda_1 \\ \lambda_1 \\ 1 \end{pmatrix} \\
(A_2^{\text{ext}})^{-1}\hat{\mathbf{X}}_2 &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{4}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{4}{\sqrt{2}} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -0.5\lambda_2 \\ (\sqrt{2}-10)\lambda_2 \\ \lambda_2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (\frac{4-21\sqrt{2}}{4})\lambda_2 + 2\sqrt{2} \\ (\frac{4-19\sqrt{2}}{4})\lambda_2 - 2\sqrt{2} \\ \lambda_2 \\ 1 \end{pmatrix}
\end{aligned}
$$

The corresponding lines in 3-D Euclidean world coordinates read:

$$
\mathbf{L_1} = \lambda_1 \begin{pmatrix} 0.5 \\ -9.5 \\ 1 \end{pmatrix} \quad , \lambda_1 \in \mathbb{R}
$$

and

$$
\mathbf{L_2} = \lambda_2 \begin{pmatrix} \frac{4-21\sqrt{2}}{4} \\ \frac{4-19\sqrt{2}}{4} \\ 1 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix} \quad , \lambda_2 \in \mathbb{R}
$$

(d) Finally, we are in the position to intersect the rays:

$$
\lambda_1 \begin{pmatrix} 0.5 \\ -9.5 \\ 1 \end{pmatrix} = \lambda_2 \begin{pmatrix} \frac{4-21\sqrt{2}}{4} \\ \frac{4-19\sqrt{2}}{4} \\ 1 \end{pmatrix} + \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix}
$$

From the last line, one can immediately see that $\lambda_1 = \lambda_2$.

$$
\lambda_1 \begin{pmatrix} \frac{21\sqrt{2}-2}{4} \\ \frac{19\sqrt{2}-34}{4} \\ 0 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ -2\sqrt{2} \\ 0 \end{pmatrix}
$$

From the fact that the values on the right have the same absolute value and those on the right haave not, one can also see directly that there is no γ_1 for which this equation holds. As a consequence, we know that the two optical rays do not intersect and we cannot compute any depth (at least not directly).

Evidently this problem resulted from a correspondence that was not estimated appropriately. However, since this is a common problem in stereo reconstruction, there is a remedy. In general, the point is taken as reconstruction that is closest to both lines in an Euclidean sense. This, however exceeds the content of this lecture.

Problem 4.

(a) The Euler-Lagrange equations for the energy functional

$$E(u,v) = \int_\Omega F(x,y,u,v,u_x,u_y,v_x,v_y)\mathrm{d}x\,\mathrm{d}y$$

are given by

$$\begin{aligned} \partial_x F_{u_x} + \partial_y F_{u_y} - F_u &= 0 \\ \partial_x F_{v_x} + \partial_y F_{v_y} - F_v &= 0 \end{aligned}$$

with boundary conditions

$$n^T \begin{pmatrix} F_{u_x} \\ F_{u_y} \end{pmatrix} = 0 \qquad n^T \begin{pmatrix} F_{v_x} \\ F_{v_y} \end{pmatrix} = 0$$

In our case the integrand is given by

$$F = \sqrt{(f_x u + f_y v + f_t)^2 + \epsilon^2} + \alpha(u_x^2 + u_y^2 + v_x^2 + v_y^2)$$

and yields the partial derivates

$$\begin{aligned} F_u &= \frac{f_x(f_x u + f_y v + f_t)}{\sqrt{(f_x u + f_y v + f_t)^2 + \epsilon^2}} \\ F_v &= \frac{f_y(f_x u + f_y v + f_t)}{\sqrt{(f_x u + f_y v + f_t)^2 + \epsilon^2}} \\ F_{u_x} &= 2\alpha u_x \\ F_{u_y} &= 2\alpha u_y \\ F_{v_x} &= 2\alpha v_x \\ F_{v_y} &= 2\alpha v_y \end{aligned}$$

Hence, the Euler-Lagrange equations are

$$\begin{aligned} 2\triangle u - \frac{f_x(f_x u + f_y v + f_t)}{\alpha\sqrt{(f_x u + f_y v + f_t)^2 + \epsilon^2}} &= 0 \\ 2\triangle v - \frac{f_y(f_x u + f_y v + f_t)}{\alpha\sqrt{(f_x u + f_y v + f_t)^2 + \epsilon^2}} &= 0 \end{aligned}$$

with boundary conditions

$$n^T \nabla u = 0 \qquad\qquad n^T \nabla v = 0$$

(b) As we have seen in the case of Horn and Schunck the Laplacean is a linear operator, since it is the sum of derivatives (differentiation is linear). However, the nonlinear factor

$$\sqrt{(f_x u + f_y v + f_t)^2 + \epsilon^2}$$

in front ot the linear expressions $f_x(f_x u + f_y v + f_t)$ and $f_y(f_x u + f_y v + f_t)$ renders the whole system of equations nonlinear.