Image Processing and Computer Vision 2005/06
Example Solutions for Theoretical Assignments 5 (T5)

Problem 1.

(a) We first compute the quartic and quintic B-splines as follows.

$$
\begin{aligned}
\beta_4 &= \int_{-\infty}^{\infty} \beta_0(x')\,\beta_3(x-x')\,dx' \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \beta_3(x-x')\,dx' \\
&= \int_{x+\frac{1}{2}}^{x-\frac{1}{2}} -\beta_3(z)\,dz \\
&= \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \beta_3(z)\,dz \\
&= \begin{cases}
0 & x < -\frac{5}{2} \\
\int_{-2}^{x+\frac{1}{2}} \frac{1}{6}(2-|z|)^3\,dz & -\frac{5}{2} \le x < -\frac{3}{2} \\
\int_{x-\frac{1}{2}}^{-1} \frac{1}{6}(2-|z|)^3\,dz + \int_{-1}^{x+\frac{1}{2}} \frac{2}{3} - z^2 + \frac{1}{2}|z|^3 dz & -\frac{3}{2} \le x < -\frac{1}{2} \\
\int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \frac{2}{3} - z^2 + \frac{1}{2}|z|^3 dz & -\frac{1}{2} \le x < \frac{1}{2} \\
\int_{x-\frac{1}{2}}^{1} \frac{2}{3} - z^2 + \frac{1}{2}|z|^3 dz + \int_{1}^{x+\frac{1}{2}} \frac{1}{6}(2-|z|)^3\,dz & \frac{1}{2} \le x < \frac{3}{2} \\
\int_{x-\frac{1}{2}}^{2} \frac{1}{6}(2-|z|)^3\,dz & \frac{3}{2} \le x < \frac{5}{2} \\
0 & x \ge \frac{5}{2}
\end{cases} \\
&= \begin{cases}
\frac{1}{24}x^4 - \frac{5}{12}|x|^3 + \frac{25}{16}x^2 - \frac{125}{48}|x| + \frac{625}{384} & \frac{3}{2} \le |x| < \frac{5}{2} \\
-\frac{1}{6}x^4 + \frac{5}{6}|x|^3 - \frac{5}{4}x^2 + \frac{5}{24}|x| + \frac{55}{96} & \frac{1}{2} < |x| < \frac{3}{2} \\
\frac{1}{4}x^4 - \frac{5}{8}x^2 + \frac{115}{192} & -\frac{1}{2} \le x \le \frac{1}{2} \\
0 & \text{else}
\end{cases}
\end{aligned}
$$

$$\beta_5 = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} \beta_4(z)\,dz$$

$$= \begin{cases} \int_{-\frac{5}{2}}^{x+\frac{1}{2}} \frac{1}{24}z^4 - \frac{5}{12}|zvert^3 + \frac{25}{16}z^2 - \frac{125}{48}|z| + \frac{625}{384}dz & -3 \le x < -2 \\ \int_{x-\frac{1}{2}}^{-\frac{3}{2}} \frac{1}{24}z^4 - \frac{5}{12}|z|^3 + \frac{25}{16}z^2 - \frac{125}{48}|z| + \frac{625}{384}dz \\ + \int_{-\frac{3}{2}}^{x+\frac{1}{2}} -\frac{1}{6}z^4 + \frac{5}{6}|z|^3 - \frac{5}{4}z^2 + \frac{5}{24}|z| + \frac{55}{96}dz & -2 \le x < -1 \\ \int_{x-\frac{1}{2}}^{-\frac{1}{2}} -\frac{1}{6}z^4 + \frac{5}{6}|z|^3 - \frac{5}{4}z^2 + \frac{5}{24}|z| + \frac{55}{96}dz \\ + \int_{-\frac{1}{2}}^{x+\frac{1}{2}} \frac{1}{4}z^4 - \frac{5}{8}z^2 + \frac{115}{192}dz & -1 \le x < 0 \\ \int_{x-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{4}z^4 - \frac{5}{8}z^2 + \frac{115}{192}dz \\ + \int_{\frac{1}{2}}^{x+\frac{1}{2}} -\frac{1}{6}z^4 + \frac{5}{6}|z|^3 - \frac{5}{4}z^2 + \frac{5}{24}|z| + \frac{55}{96}dz & 0 \le x < 1 \\ \int_{x-\frac{1}{2}}^{\frac{3}{2}} -\frac{1}{6}z^4 + \frac{5}{6}|z|^3 - \frac{5}{4}z^2 + \frac{5}{24}|z| + \frac{55}{96}dz \\ + \int_{\frac{3}{2}}^{x+\frac{1}{2}} \frac{1}{24}z^4 - \frac{5}{12}|z|^3 + \frac{25}{16}z^2 - \frac{125}{48}|z| + \frac{625}{384} & 1 \le x < 2 \\ \int_{x-\frac{1}{2}}^{\frac{5}{2}} \frac{1}{24}z^4 - \frac{5}{12}|z|^3 + \frac{25}{16}z^2 - \frac{125}{48}|z| + \frac{625}{384}dz & 2 \le x < 3 \\ 0 & \text{else} \end{cases}$$

$$= \begin{cases} -\frac{1}{120}|x|^5 + \frac{1}{8}x^4 - \frac{3}{4}|x|^3 + \frac{9}{4}x^2 - \frac{27}{8}|x| + \frac{81}{40} & 2 \le |x| \le 3 \\ \frac{1}{24}|x|^5 - \frac{3}{8}x^4 + \frac{5}{4}|x|^3 - \frac{7}{4}x^2 + \frac{5}{8}|x| + \frac{17}{40} & 1 \le |x| \le 2 \\ -\frac{1}{12}|x|^5 + \frac{1}{4}x^4 - \frac{1}{2}x^2 + \frac{11}{20} & -1 \le x < 1 \\ 0 & \text{else} \end{cases}$$

Hence, we get

$$\beta_5 = \begin{cases} \frac{11}{20} & x = 0 \\ \frac{13}{60} & |x| = 1 \\ \frac{1}{120} & |x| = 2 \end{cases}$$

Thus the interpolation condition is described by the following linear system of equations

$$
\begin{pmatrix}
\frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
\frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 & \cdots & \cdots & \cdots & 0 \\
 & \ddots & \ddots & \ddots & \ddots & \ddots & & & & \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\
0 & \cdots & \cdots & \cdots & 0 & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} \\
0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{120} & \frac{13}{60} & \frac{11}{20}
\end{pmatrix}
\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ c_{N-1} \\ c_n \end{pmatrix}
=
\begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_{N-1} \\ f_n \end{pmatrix}
$$

(b) The linear system can be solved efficiently using the simple Gauß algorithm (Thomas Algorithm). This is because the matrix is sparse (pentadiagonal) and the diagonals lie dense (not separated by zeros). Thus, no information will be filled in when using the standard Gauß method.

(c) Quartic B-splines yield a similiar linear system of equations since we also get five non-vanishing function values for β_4. Hence while solving the system requires the same cost, the interpolation quality is worse than interpolation based on quintic B-splines.

Problem 2.

(a) The Taylor expansion of f in x_i is given by

$$f(x) = f_i + f_i^{(1)}(x - x_i) + \frac{f_i^{(2)}}{2}(x - x_i)^2 + \frac{f_i^{(3)}}{6}(x - x_i)^3 + \frac{f_i^{(4)}}{24}(x - x_i)^4 + O(x^5).$$

Plugging in the points of our given signal values yields

$$\begin{aligned}
f_{i-2} &= f_i - 2hf_i^{(1)} + 2h^2 f_i^{(2)} - \frac{4}{3}h^3 f_i^{(3)} + \frac{2}{3}h^4 f_i^{(4)} + O(h^5) \\
f_{i-1} &= f_i - hf_i^{(1)} + \frac{1}{2}h^2 f_i^{(2)} - \frac{1}{6}h^3 f_i^{(3)} + \frac{1}{24}h^4 f_i^{(4)} + O(h^5) \\
f_i &= f_i \\
f_{i+1} &= f_i + hf_i^{(1)} + \frac{1}{2}h^2 f_i^{(2)} + \frac{1}{6}h^3 f_i^{(3)} + \frac{1}{24}h^4 f_i^{(4)} + O(h^5) \\
f_{i+2} &= f_i + 2hf_i^{(1)} + 2h^2 f_i^{(2)} + \frac{4}{3}h^3 f_i^{(3)} + \frac{2}{3}h^4 f_i^{(4)} + O(h^5)
\end{aligned}$$

The approximation we are looking for shall be of the form

$$f_i^{(1)} \approx \alpha f_{i-2} + \beta f_{i-1} + \gamma f_i + \delta f_{i+1} + \epsilon f_{i+2}$$

where $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ are the coefficicients we are interested in.

Plugging in the Taylor approximations yields

$$\begin{aligned}
f_i^{(1)} \approx\ & \alpha \left(f_i - 2hf_i^{(1)} + 2h^2 f_i^{(2)} - \frac{4}{3}h^3 f_i^{(3)} + \frac{2}{3}h^4 f_i^{(4)} \right) \\
& + \beta \left(f_i - hf_i^{(1)} + \frac{1}{2}h^2 f_i^{(2)} - \frac{1}{6}h^3 f_i^{(3)} + \frac{1}{24}h^4 f_i^{(4)} \right) \\
& + \gamma f_i \\
& + \delta \left(f_i + hf_i^{(1)} + \frac{1}{2}h^2 f_i^{(2)} + \frac{1}{6}h^3 f_i^{(3)} + \frac{1}{24}h^4 f_i^{(4)} \right) \\
& + \epsilon \left(f_i + 2hf_i^{(1)} + 2h^2 f_i^{(2)} + \frac{4}{3}h^3 f_i^{(3)} + \frac{2}{3}h^4 f_i^{(4)} \right) \\
=\ & (\alpha + \beta + \gamma + \delta + \epsilon) f_i + (-2\alpha - \beta + \delta + 2\epsilon) h f_i^{(1)} \\
& + \left(2\alpha + \frac{1}{2}\beta + \frac{1}{2}\delta + 2\epsilon \right) h^2 f_i^{(2)} + \left(-\frac{4}{3}\alpha - \frac{1}{6}\beta + \frac{1}{6}\delta + \frac{4}{3}\epsilon \right) h^3 f_i^{(3)} \\
& + \left(\frac{2}{3}\alpha + \frac{1}{24}\beta + \frac{1}{24}\delta + \frac{2}{3}\epsilon \right) h^4 f_i^{(4)}
\end{aligned}$$

Comparing coefficients gives the linear system of equations

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} & 2 \\ -\frac{4}{3} & -\frac{1}{6} & 0 & \frac{1}{6} & \frac{4}{3} \\ \frac{2}{3} & \frac{1}{24} & 0 & \frac{1}{24} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \\ \epsilon \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{h} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

which must be satisfied by the solution.

(b) Obviously the given vector is a solution of the linear system of equations above. For determining the order of consistency of this approximation, we use the longer Taylor expansion:

$$f(x) = f_i + f_i^{(1)}(x-x_i) + \frac{f_i^{(2)}}{2}(x-x_i)^2 + \frac{f_i^{(3)}}{6}(x-x_i)^3 + \frac{f_i^{(4)}}{24}(x-x_i)^4 + \frac{f_i^{(5)}}{120}(x-x_i)^5 + O(x^6).$$

Which gives the approximations

$$\begin{aligned}
f_{i-2} &= f_i - 2hf_i^{(1)} + 2h^2 f_i^{(2)} - \frac{4}{3}h^3 f_i^{(3)} + \frac{2}{3}h^4 f_i^{(4)} - \frac{4}{15}h^5 f_i^{(5)} + O(h^6) \\
f_{i-1} &= f_i - hf_i^{(1)} + \frac{1}{2}h^2 f_i^{(2)} - \frac{1}{6}h^3 f_i^{(3)} + \frac{1}{24}h^4 f_i^{(4)} - \frac{1}{120}h^5 f_i^{(5)} + O(h^6) \\
f_i &= f_i \\
f_{i+1} &= f_i + hf_i^{(1)} + \frac{1}{2}h^2 f_i^{(2)} + \frac{1}{6}h^3 f_i^{(3)} + \frac{1}{24}h^4 f_i^{(4)} + \frac{1}{120}h^5 f_i^{(5)} + O(h^6) \\
f_{i+2} &= f_i + 2hf_i^{(1)} + 2h^2 f_i^{(2)} + \frac{4}{3}h^3 f_i^{(3)} + \frac{2}{3}h^4 f_i^{(4)} + \frac{4}{15}h^5 f_i^{(5)} + O(h^6)
\end{aligned}$$

Using these approximations and $f_i^{(1)} \approx \frac{1}{12h}\left(f_{i-2} - 8f_{i-1} + 8f_{i+1} - f_{i+2}\right)$ we obtain

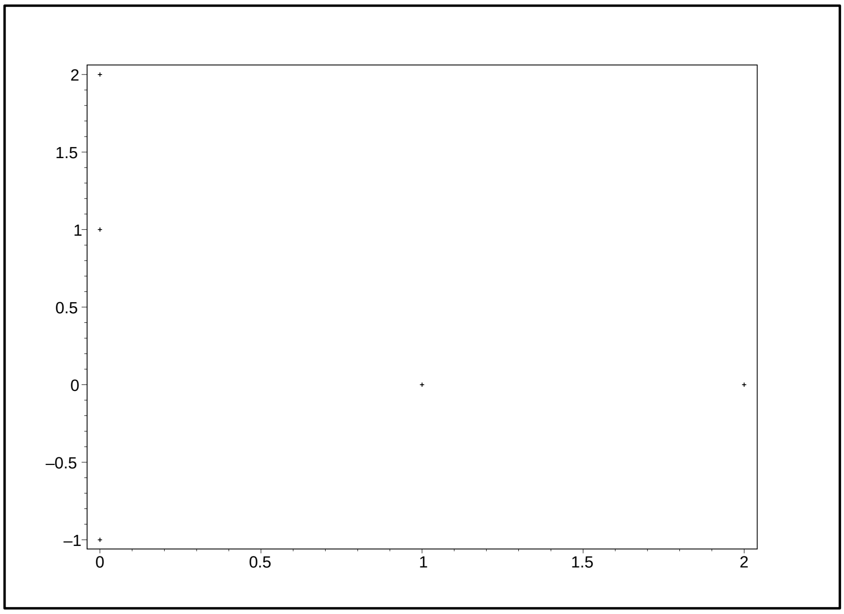
$$\begin{aligned}
f_i^{(1)} &\approx \frac{1}{12h}\left[(1-8+8-1)\, f_i \right.\\
&\quad + (-2+8+8-2)\, hf_i^{(1)} \\
&\quad + (2-4+4-2)\, h^2 f_i^{(2)} \\
&\quad + \left(-\frac{4}{3} + \frac{4}{3} + \frac{4}{3} - \frac{4}{3}\right) h^3 f_i^{(3)} \\
&\quad + \left(\frac{2}{3} - \frac{1}{3} + \frac{1}{3} - \frac{2}{3}\right) h^4 f_i^{(4)} \\
&\quad \left. + \left(-\frac{4}{15} + \frac{1}{15} + \frac{1}{15} - \frac{4}{15}\right) h^5 f_i^{(5)} + O(h^6)\right] \\
&= f_i^{(1)} - \frac{1}{30}h^4 + O(h^5)
\end{aligned}$$

Therefore, the order of consistency for this approximation is 4.

(c) For approximating a derivative of order d with $n > d$ points, the coefficients in front of $h^i f_i^{(i)}$ for all $i < n, i \neq d$ always vanish. Therefore, the order of consistency must be at least $n - d$. Note that this is just a lower bound, the order of consistency could be higher.
Example 1: In (b), we approximated a derivative of order one using five points, therefore the order of consistency must be at least 4, which is the case here.
Example 2: When approximating $f_i^{(2)}$ with five points $f_{i-2}, f_{i-1}, f_i, f_{i+1}, f_{i+2}$ one gets the filter mask $\frac{1}{h}\left(-\frac{1}{12} + \frac{4}{3} - \frac{5}{2} + \frac{4}{3} - \frac{1}{12}\right)$. In this case, the order of consistency must be at least $5 - 2 = 3$. Actually, this approximation has, however, the order of consistency 4.

Problem 3.

(a) We consider the set of point set $\{(1,0),(2,0),(0,1),(0,2),(0,-1)\}$.



With the help of the Hough transform we want to determine which three of these points belong to the same line. We remember that the parameter space has the coordinates (Φ, d), and each line in the image

$$\{(x,y) \mid d = \cos\Phi + y\sin\Phi\}$$

can be represented as a point (Φ, d) in this space. Vice versa, for each point (x, y) we can write down all lines which the point may belong to. This yields the trigonometric curve

$$d = x\cos\Phi + y\sin\Phi$$

in the parameter space. (In this case, x and y are fixed.)

For the point set given above this leads to the five trigonometric curves:

1. point (1,0): $d = \cos\phi$
2. point (2,0): $d = 2\cos\phi$
3. point (0,1): $d = \sin\phi$
4. point (0,2): $d = 2\sin\phi$

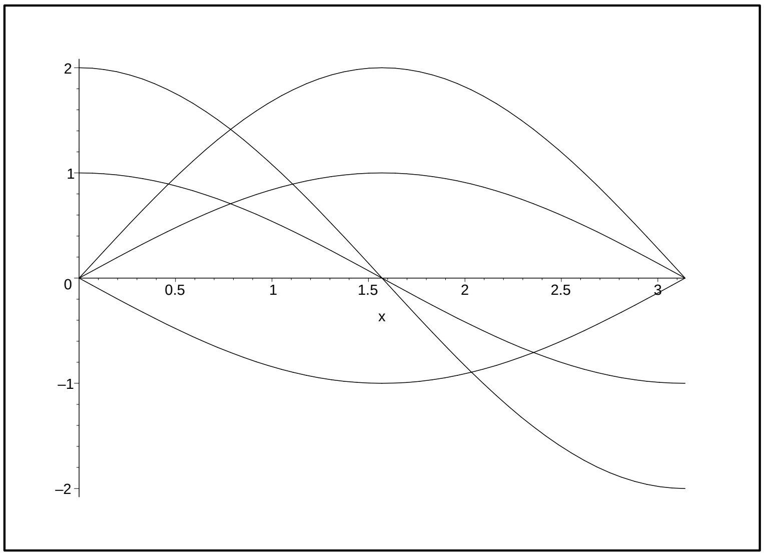
5. point (0,-1): $d = -\sin\phi$

Two points belonging to the same line lead to an intersection point of the corresponding trigonometric curves in the parameter space. Since for each pair of points there is one line they both belong to, only the intersection of three or more trigonometric curves is interesting. From the exercise sheet we know that we have to search an intersection of tree curves. For symmetry and periodicticy reasons we only need to consider an arbitrary interval of length Π (here [0,Π)). We see that the curves (1) and (2) con only have the same value at points where the cosine vanishes (x=Π/2), similary (1)-(3) can only intersect if the sinus funktion vanishes (x=0).

We moreover see that there is exactly one point where tree curves intersect, namely the curves 3,4 and 5 in the point (0,0). Thus a discretisation of the parameter space is not necessary in this simple example. The point (0,0) coresponds to the y-axis:

$$0 = x\cos(0) + y\sin(0) = x * 1 + y * 0 = x$$

Remark: When discretising the algorithm, care must be taken in order not to lose intersection points. In the discrete space we mark each point lying on a trigonometric curve, and finally choose the cell with most marks. Due to the discretisation it is possible that not all intersection points are found.



The figure shows the 5 functions. We can see the one point where three curves intersect is (0,0). (The point $(0, \Phi)$ is not in our interval and leads also to the funktion x=0.)

(b) Now we have to apply the Hough transform for finding a circle. The points are the same as in part a.

The equation for a circle with centre (a,b) and radius r is:

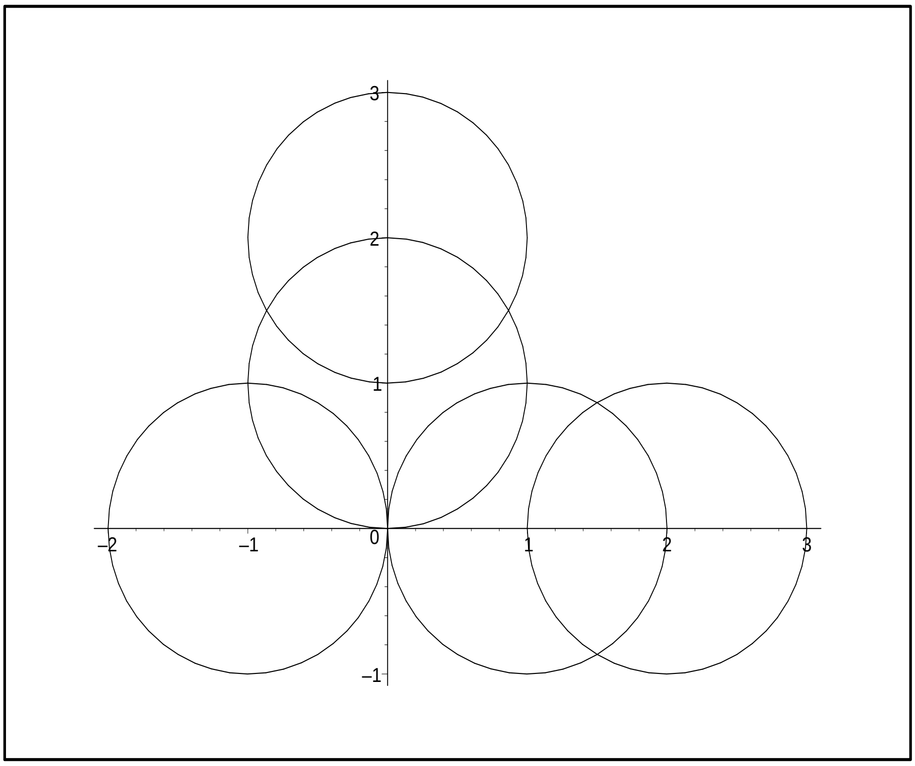
$$|x-a|^2 + |y-b|^2 - r^2 = 0$$

Here, r=1. So we have

$$|x-a|^2 + |y-b|^2 = 1.$$

For the point set given above, this leads to the five curves:

1. point (1,0): $(1-a)^2 + b^2 = 1 \quad \rightarrow a = \pm\sqrt{(1-b^2)} + 1$
2. point (2,0): $(2-a)^2 + b^2 = 1 \quad \rightarrow a = \pm\sqrt{(1-b^2)} + 2$
3. point (0,1): $a^2 + (1-b)^2 = 1 \quad \rightarrow a = \pm\sqrt{(1-(1-b)^2)}$
4. point (0,2): $a^2 + (2-b)^2 = 1 \quad \rightarrow a = \pm\sqrt{(1-(2-b)^2)}$
5. point (0,-1): $a^2 + (-1-b)^2 = 1 \quad \rightarrow a = \pm\sqrt{(1-(-1-b)^2)}$



That are 5 circles with radius 1. The circles (1),(3) and (5) intersect in the point (0,0). This means the points 1, 3, 5 belong to the same circle with centre (0,0) and radius 1.

Problem 4.

(a) Since we consider a quadratic form, this implies that J is not only postive semidefinite but also symmetric. This property can be used to perform an eigenvalue decomposition of J and thus to rewrite the quadratic form as

$$\mathbf{x}^\top J\mathbf{x} = \mathbf{x}^\top W\Lambda W^\top \mathbf{x}.$$

Here, W is an orthonormal $m \times m$ matrix with the normalised eigenvectors of J as columns and Λ is a diagonal $m \times m$ matrix with the eigenvalues of J given by $\lambda_1, .., \lambda_m$ as entries. Since W is an orthonormal matrix (rotation matrix), we can simplify our minimisation problem by substituting the vector

$$\mathbf{v} = W^\top \mathbf{x}.$$

One should note the new vector $\mathbf{v}$ has the same length as the old one $\mathbf{x}$ (lenghts are preserved under rotation). In the modified (rotated) coordinate system, the minimisation problem is now given by

$$\mathbf{v}^\top \Lambda \mathbf{v}$$

where the same side constraint holds as for the original problem, namely $||\mathbf{v}|| = 1$ (due to the preservation of lenghts). Without loss of generality, let us assume that λ_m is the smallest eigenvalue. Then, the following estimate holds:

$$\begin{aligned}
\mathbf{v}^\top \Lambda \mathbf{v} &= \sum_{i=1}^{m} \lambda_i v_i^2 \\
&\geq \sum_{i=1}^{m} \lambda_m v_i^2 \\
&= \lambda_m \sum_{i=1}^{m} v_i^2 \\
&= \lambda_m ||\mathbf{v}||^2 \\
&= \lambda_m .
\end{aligned}$$

Evidently, this estimate forms a lower bound for the result of the minimisation problem. Thus, if the equality holds, the whole expression is minimised. This in turn is the case if

$$\mathbf{v} = \mathbf{v}_{min} = (0, ...0, 1)^\top .$$

Let us now transform our solution back to orginal (non-rotated) coordinate system. Then, we get

$$\begin{aligned}
\mathbf{x}_{min} &= W\mathbf{v}_{min} \\
&= \mathbf{w}_m ,
\end{aligned}$$

which is the m-th column of matrix W, i.e. the eigenvector associated to the (smallest) eigenvalue λ_m.

(b) In the case $\rho = 0$, the structure tensor comes down to

$$\begin{aligned} J_\rho &= K_\rho * \left(\nabla f \, \nabla f^\top\right) \\ &\overset{\rho=0}{=} K_0 * \left(\nabla f \, \nabla f^\top\right) \\ &= \nabla f \, \nabla f^\top. \end{aligned}$$

By performing an eigenvalue decomposition of J_0 we see that

$$J_0 = \begin{pmatrix} \frac{f_x}{||\nabla f||} & \frac{f_y}{||\nabla f||} \\ \frac{f_y}{||\nabla f||} & \frac{-f_x}{||\nabla f||} \end{pmatrix} \begin{pmatrix} ||\nabla f||^2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{f_x}{||\nabla f||} & \frac{f_y}{||\nabla f||} \\ \frac{f_y}{||\nabla f||} & \frac{-f_x}{||\nabla f||} \end{pmatrix}^\top .$$

While the eigenvalue in direction of the gradient ∇f is given by $||\nabla f||^2$, the eigenvalue orthogonal to it is exactly 0. This is not surprising, since in the special case $\rho = 0$ the structure tensor is not averaged over a neighbourhood: It is only constructed from one gradient, namely ∇f at the current position (x, y). Since one eigenvalue is zero, no no corners can be detected. However, the detection of edges is still possible: This only requires one eigenvalue to be different from zero.