Image Processing and Computer Vision 2005/06
Example Solutions for Theoretical Assignments 4 (T4)

Problem 1.

To show convexity of E we chose u,v $\in \mathbb{R}^N$ such that $u \neq v$, and $0 \leq \beta \leq 1$. We have then to show that:

$$E_f(\beta u + (1-\beta)v) < \beta E_f(u) + (1-\beta)E_f(v)$$

The functions $g(u) = u^2$ and $h_a(u) = (u-a)^2$ are strictly convex for all $x, a \in \mathbb{R}$ since $g(u)'' = 2 > 0$ and $h_a(u)'' = 2 > 0$

$\Rightarrow (\beta u + (1-\beta)v)^2 < \beta u^2 + (1-\beta)v^2$ and
$\Rightarrow ((\beta u + (1-\beta)v - a))^2 < \beta(u-a)^2 + (1-\beta)(v-a)^2$

$$E_f(u) = \frac{1}{2}\sum_{k=1}^{N}(u_k - f_k)^2 + \frac{a}{2}\sum_{k=1}^{N-1}(u_{k+1} - u_k)^2$$

$$E_f(\beta u + (1-\beta)v)$$

$$= \frac{1}{2}\sum_{k=1}^{N}(\beta u_k + (1-\beta)v_k - f_k)^2$$

$$+\frac{a}{2}\sum_{k=1}^{N-1}((\beta u_{k+1} + (1-\beta)v_{k+1}) - (\beta u_k + (1-\beta)v_k))^2$$

$$= \frac{1}{2}\sum_{k=1}^{N}(\beta u_k + (1-\beta)v_k - f_k)^2$$

$$+\frac{a}{2}\sum_{k=1}^{N-1}(\beta(u_{k+1} - u_k) + (1-\beta)(v_{k+1} - v_k))^2$$

(because of the convexity of u^2 and $(u-a)^2$)

$$< \frac{1}{2}\sum_{k=1}^{N}(\beta(u_k - f_k)^2 + (1-\beta)(v_k - f_k)^2)$$

$$+\frac{a}{2}\sum_{k=1}^{N-1}(\beta(u_{k+1}-u_k)^2+(1-\beta)(v_{k+1}-v_k)^2)$$

$$=\frac{1}{2}\beta\sum_{k=1}^{N}(u_k-f_k)^2+\frac{1}{2}(1-\beta)\sum_{k=1}^{N}(v_k-f_k)^2$$

$$+\frac{a}{2}\beta\sum_{k=1}^{N-1}(u_{k+1}-u_k)^2+\frac{a}{2}(1-\beta)\sum_{k=1}^{N-1}(v_{k+1}-v_k)^2$$

$$=\beta E_f(u)+(1-\beta)E_f(v)$$

We see E is strictly convex. The consequence: if a solution u exists, so that $\nabla E(u)=0$, then u is a minimum and this minimum is unique.

Problem 2.

$$A=\begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix} \qquad b=\begin{pmatrix} 7 \\ 9 \\ 6 \end{pmatrix}$$

(a) Determine LR - decomposition

$$L=\begin{pmatrix} 1 & 0 & 0 \\ l_1 & 1 & 0 \\ 0 & l_2 & 1 \end{pmatrix} \quad \text{and} \quad R=\begin{pmatrix} m_1 & r_1 & 0 \\ 0 & m_2 & r_2 \\ 0 & 0 & m_3 \end{pmatrix}$$

such that $AL=R$.

Let us denote the entries of A by

$$A=\begin{pmatrix} 10 & -1 & 0 \\ -1 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix}=\begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \gamma_1 & \alpha_2 & \beta_2 \\ 0 & \gamma_2 & \alpha_3 \end{pmatrix}$$

Evidently, r_i=β_i for $i=1,2$ so we obtain $r_1=-1$ and $r_2=-2$. Moreover, we know that $m_1=\alpha_1=10$.

Since $l_i = \frac{\gamma_i}{m_i}$ and $m_{i+1} = \alpha_{i+1} - l_i\beta_i$ for $i = 1, 2$ we can compute furthermore

$$l_1 = \frac{\gamma_1}{m_1} = -\frac{1}{10}, \qquad m_2 = \alpha_2 - l_1\beta_1 = 10 + \frac{1}{10}(-1) = \frac{99}{10},$$

$$l_2 = \frac{\gamma_2}{m_2} = \frac{-2}{\frac{99}{10}} = -\frac{20}{99}, \qquad m_3 = \alpha_3 - l_2\beta_2 = 10 + \frac{20}{99}(-2) = \frac{950}{99}.$$

Thus, we get

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{10} & 1 & 0 \\ 0 & -\frac{20}{99} & 1 \end{pmatrix} \quad \text{and} \quad R = \begin{pmatrix} 10 & -1 & 0 \\ 0 & \frac{99}{10} & -2 \\ 0 & 0 & \frac{950}{99} \end{pmatrix}.$$

(b) Solve $Au = b$ via forward elimination and backward substitution.

- Forward elimination: Solve $Ly = b$.
  We know that $y_1 = b_1 = 7$ and that $y_i = b_i - l_{i-1}y_{i-1}$ for $i = 2, 3$.
  Thus, we obtain

  $$y_2 = b_2 - l_1 y_1 = 9 + \frac{1}{10}\,7 = \frac{97}{10},$$
  $$y_3 = b_3 - l_2 y_2 = 6 + \frac{20}{99}\,\frac{97}{10} = \frac{788}{99},$$

  which gives

  $$y = \begin{pmatrix} 7 \\ \frac{97}{10} \\ \frac{788}{99} \end{pmatrix}.$$

- Backward substitution: Solve $Ru = y$.
  We know that $u_3 = \frac{y_3}{m_3} = \frac{394}{475}$ and that $u_i = \frac{y_i - \beta_i u_{i+1}}{m_i}$ for $i = 2, 1$. Thus, we obtain

  $$u_2 = \frac{y_2 - \beta_2 u_3}{m_2} = \frac{545}{475},$$
  $$u_1 = \frac{y_1 - \beta_1 u_2}{m_1} = \frac{387}{475},$$

  which gives

  $$u = \begin{pmatrix} \frac{387}{475} \\ \frac{109}{95} \\ \frac{394}{475} \end{pmatrix}.$$

(c) Compute the first two iterations of the Jacobi method. Starting vector is $x^{(0)} = (0,0,0)^\top$.

Idea of Jacobi method: Decompose a such that $A = D - N$, where D is the diagonal part and N is the rest. So in our case we have

$$D = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

The Jacobi method then reads:

$$x^{(k+1)} = D^{-1}(Nx^{(k)} + b)$$

- 1. step: k=0

$$\begin{aligned} x^{(1)} &= D^{-1}(Nx^{(0)} + b) \\ &= \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 7 \\ 9 \\ 6 \end{pmatrix} \right] \\ &= \begin{pmatrix} \frac{7}{10} \\ \frac{9}{10} \\ \frac{6}{10} \end{pmatrix} \end{aligned}$$

- 2. step: k=1

$$\begin{aligned} x^{(2)} &= D^{-1}(Nx^{(1)} + b) \\ &= \begin{pmatrix} \frac{1}{10} & 0 & 0 \\ 0 & \frac{1}{10} & 0 \\ 0 & 0 & \frac{1}{10} \end{pmatrix} \left[\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} \frac{7}{10} \\ \frac{9}{10} \\ \frac{6}{10} \end{pmatrix} + \begin{pmatrix} 7 \\ 9 \\ 6 \end{pmatrix} \right] \\ &= \begin{pmatrix} \frac{79}{100} \\ \frac{109}{10} \\ \frac{39}{50} \end{pmatrix} \end{aligned}$$

$$x^{(2)} = \begin{pmatrix} 0.79 \\ 1.09 \\ 0.78 \end{pmatrix} \qquad u \approx \begin{pmatrix} 0.8147 \\ 1.1474 \\ 0.8295 \end{pmatrix}$$

Comparing $x^{(2)}$ with the correct solution u, one sees that the Jacobi method already yield a relatively good approximation of u after only two iterations.

Problem 3.

(a) For the integrand

$$F(x,u,u') := \left(\frac{1}{2}(u-f)^2 + \alpha\lambda^2\sqrt{1+\frac{(u')^2}{\lambda^2}}\right) \mathrm{dx}$$

we obtain the partial derivatives

$$\begin{aligned} F_u &= u - f \\ F_{u'} &= \frac{\alpha}{\sqrt{1+\frac{(u')^2}{\lambda^2}}} u' \end{aligned}$$

This implies the Euler-Lagrange equation

$$\begin{aligned} 0 &= u - f - \alpha \frac{d}{dx} \frac{1}{\sqrt{1+\frac{(u')^2}{\lambda^2}}} u' \\ &= u - f - \alpha \left(u' \frac{d}{dx} \frac{1}{\sqrt{1+\frac{(u')^2}{\lambda^2}}} + u'' \frac{1}{\sqrt{1+\frac{(u')^2}{\lambda^2}}} \right) \\ &= u - f - \alpha \left(-(u')^2 u'' \frac{1}{\lambda^2\left(1+\frac{(u')^2}{\lambda^2}\right)^{\frac{3}{2}}} + u'' \frac{1}{\sqrt{1+\frac{(u')^2}{\lambda^2}}} \right) \\ &= u - f - \alpha u'' \left(\frac{1}{\sqrt{1+\frac{(u')^2}{\lambda^2}}} - (u')^2 \frac{1}{\lambda^2\left(1+\frac{(u')^2}{\lambda^2}\right)^{\frac{3}{2}}} \right) \\ &= u - f - \alpha u'' \frac{1}{\left(1+\frac{(u')^2}{\lambda^2}\right)^{\frac{3}{2}}} \end{aligned}$$

with boundary conditions:

$$F_{u'} = 0$$

in $x = a$ and $x = b$

(b) λ is a contrast parameter: Locations with $|\nabla u| > \lambda$ are regarded as edges. There, the diffusivity is reduced significantly.

(c) First, we show that the function f

$$f : x \mapsto \sqrt{1 + \left(\frac{x}{\lambda}\right)^2}$$

is strictly convex. We substitute $y := \frac{x}{\lambda}$ and get the function g:

$$g : y \mapsto \sqrt{1 + y^2}$$

The derivatives of this function are:

$$\frac{d}{dy} g = \frac{y}{\sqrt{1 + y^2}}$$

$$\frac{d^2}{(dy)^2} g = \frac{1}{(1 + y^2)^{\frac{3}{2}}}$$

As one can see, the second derivative is strictly positive everywhere.

We also know that $x \mapsto x^2$ is strictly convex. Thus:

$$\begin{aligned}
E_f(\beta u+(1-\beta)v) &:= \frac{1}{2}\int_a^b (\beta u+(1-\beta)v-f)^2 \mathrm{dx} \\
&\quad +\alpha\lambda^2\int_a^b \sqrt{1+\frac{(\beta u'+(1-\beta)v')^2}{\lambda^2}}\mathrm{dx} \\
&= \frac{1}{2}\int_a^b (\beta(u-f)+(1-\beta)(v-f))^2 \mathrm{dx} \\
&\quad +\alpha\lambda^2\int_a^b \sqrt{1+\frac{(\beta u'+(1-\beta)v')^2}{\lambda^2}}\mathrm{dx} \\
&\quad \text{Using the strict convexity of } x \mapsto x^2 \text{ we get} \\
&< \frac{\beta}{2}\int_a^b (u-f)^2\mathrm{dx} + \frac{1-\beta}{2}\int_a^b (v-f)^2\mathrm{dx} \\
&\quad +\alpha\lambda^2\int_a^b \sqrt{1+\frac{(\beta u'+(1-\beta)v')^2}{\lambda^2}}\mathrm{dx} \\
&\quad \text{Using the strict convexity of } x \mapsto \sqrt{1+\left(\frac{x}{\lambda}\right)^2} \text{ we obtain} \\
&< \frac{\beta}{2}\int_a^b (u-f)^2\mathrm{dx} + \frac{1-\beta}{2}\int_a^b (v-f)^2\mathrm{dx} \\
&\quad +\alpha\beta\lambda^2\int_a^b \sqrt{1+\frac{u'^2}{\lambda^2}}\mathrm{dx} \\
&\quad +\alpha(1-\beta)\lambda^2\int_a^b \sqrt{1+\frac{v'^2}{\lambda^2}}\mathrm{dx} \\
&= \beta E_f(u)+(1-\beta)E_f(v)
\end{aligned}$$

Therefore this energy functional is strictly convex. So the solution of the system is unique, and it is a minimizer.

Problem 4.

(a) The discrete analogue is given by:

$$E_f(u) := \frac{1}{2}\sum_{k=1}^{N}(u_k-f_k)^2 + \alpha\lambda^2\sum_{k=1}^{N-1}\sqrt{1+\frac{(u_{k+1}-u_k)^2}{h^2\lambda^2}}$$

where the signal is given as $[u_1, u_2, \ldots, u_N]$.

(b) The first partial derivatives w.r.t. $u_1, \ldots, u_N$ must vanish, i.e.:

$$
\begin{aligned}
0 &= \frac{\delta E_f}{\delta u_1} = u_1 - f_1 - \frac{\alpha}{h^2}\frac{u_2 - u_1}{\sqrt{1 + \frac{(u_2-u_1)^2}{h^2\lambda^2}}} \\
0 &= \frac{\delta E_f}{\delta u_i} = u_i - f_i + \frac{\alpha}{h^2}\left(\frac{u_i - u_{i-1}}{\sqrt{1 + \frac{(u_i-u_{i-1})^2}{h^2\lambda^2}}} - \frac{u_{i+1} - u_i}{\sqrt{1 + \frac{(u_{i+1}-u_i)^2}{h^2\lambda^2}}}\right)\ ,\ i = 2, ..., N-1 \\
0 &= \frac{\delta E_f}{\delta u_N} = u_N - f_N + \frac{\alpha}{h^2}\frac{u_N - u_{N-1}}{\sqrt{1 + \frac{(u_N-u_{N-1})^2}{h^2\lambda^2}}}
\end{aligned}
$$