Image Processing and Computer Vision 2005/06
Example Solutions for Theoretical Assignments 3 (T3)

Problem 1.

Note that we have to distinguish between x- and y-direction since the filters may have different effects in each direction.

(a)

- x-direction: scaled box filter $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ ⇒lowpass
- y-direction: Gaussian approximated by $\frac{1}{4}\begin{pmatrix} 1 & 2 & 1 \end{pmatrix}$ ⇒lowpass

(b)

- x-direction and y-direction: difference between indenty and lowpass ⇒ highpass

$$\frac{1}{16}\begin{pmatrix} -1 & -2 & -1 \\ -2 & 12 & -2 \\ -1 & -2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{16}\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}$$

(c)

- x-direction: as before we can write the filter as the difference between identity and lowpass filter ⇒highpass

$$\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -1 & 2 & -1 \end{pmatrix}$$

- y-direction: an average filter (box filter) ⇒ lowpass

(d)

- x-direction + y-direction: filter can be written as the difference between two lowpass filters ⇒bandpass

$$\begin{pmatrix} -1 & -4 & -6 & -4 & -1 \\ -4 & 0 & 8 & 0 & -4 \\ -6 & 8 & 28 & 8 & -6 \\ -4 & 0 & 8 & 0 & -4 \\ -1 & -4 & -6 & -4 & -1 \end{pmatrix} = \frac{1}{256}\left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 32 & 16 & 0 \\ 0 & 32 & 64 & 32 & 0 \\ 0 & 16 & 32 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 6 & 4 & 1 \\ 4 & 16 & 24 & 16 & 4 \\ 6 & 24 & 36 & 24 & 6 \\ 4 & 16 & 24 & 16 & 4 \\ 1 & 4 & 6 & 4 & 1 \end{pmatrix}\right)$$

Problem 2.

Starting from $c_{0,k} = f_k$ for $k = 0, \dots, 7$ and $f = (4, 2, -4, 0, 2, -1, -4, -3)$, we compute

$$c_{j,k} = \frac{1}{\sqrt{2}}\left(c_{j-1,2k} + c_{j-1,2k+1}\right) \qquad d_{j,k} = \frac{1}{\sqrt{2}}\left(c_{j-1,2k} - c_{j-1,2k+1}\right)$$

for the scales $j = 1, 2, 3$:

$$\begin{array}{lcl|lcl}
c_{1,0} = \frac{1}{\sqrt{2}}(4+2) & = & 3\sqrt{2} & d_{1,0} = \frac{1}{\sqrt{2}}(4-2) & = & \sqrt{2} \\
c_{1,1} = \frac{1}{\sqrt{2}}(-4+0) & = & -2\sqrt{2} & d_{1,1} = \frac{1}{\sqrt{2}}(-4-0) & = & -2\sqrt{2} \\
c_{1,2} = \frac{1}{\sqrt{2}}(2+(-1)) & = & \frac{1}{2}\sqrt{2} & d_{1,2} = \frac{1}{\sqrt{2}}(2-(-1)) & = & \frac{3}{2}\sqrt{2} \\
c_{1,3} = \frac{1}{\sqrt{2}}(-4+(-3)) & = & -\frac{7}{2}\sqrt{2} & d_{1,3} = \frac{1}{\sqrt{2}}(-4-(-3)) & = & -\frac{1}{2}\sqrt{2} \\
c_{2,0} = \frac{1}{\sqrt{2}}\left(3\sqrt{2}+(-2\sqrt{2})\right) & = & 1 & d_{2,0} = \frac{1}{\sqrt{2}}\left(3\sqrt{2}-(-2\sqrt{2})\right) & = & 5 \\
c_{2,1} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\sqrt{2}+(-\frac{7}{2}\sqrt{2})\right) & = & -3 & d_{2,1} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\sqrt{2}-(-\frac{7}{2}\sqrt{2})\right) & = & 4 \\
c_{3,0} = \frac{1}{\sqrt{2}}(1+(-3)) & = & -\sqrt{2} & d_{3,0} = \frac{1}{\sqrt{2}}(1-(-3)) & = & 2\sqrt{2}
\end{array}$$

Thus, the transformed signal is

$$\hat{f} = (-\sqrt{2}, 2\sqrt{2}, 5, 4, \sqrt{2}, -2\sqrt{2}, \frac{3}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$$

In order to remove exactly three parameters, the threshold parameter T for the hard wavelet shrinkage has to be chosen such that $\frac{3}{2}\sqrt{2} \le T < 2\sqrt{2}$. (Because the four smallest absolute values of the transformed signal (without the scaling coefficient) are $\frac{1}{2}\sqrt{2}$, $\sqrt{2}$, $\frac{3}{2}\sqrt{2}$ and $2\sqrt{2}$).

After performing hard shrinkage with such a T, the signal is:

$$\hat{f}' = (-\sqrt{2}, 2\sqrt{2}, 5, 4, 0, -2\sqrt{2}, 0, 0)$$

Remember that the scaling coefficient is not changed. To transform the signal back, we compute

$$c_{j,2k} = \frac{1}{\sqrt{2}}\left(c_{j+1,k} + d_{j+1,k}\right) \qquad c_{j,2k+1} = \frac{1}{\sqrt{2}}\left(c_{j+1,k} - d_{j+1,k}\right)$$

for $j = 2, 1, 0$:

$$\begin{array}{ll|ll} c_{2,0} = \frac{1}{\sqrt{2}}\left(-\sqrt{2} + 2\sqrt{2}\right) = & 1 & c_{1,0} = \frac{1}{\sqrt{2}}(1+5) = & 3\sqrt{2} \\ c_{2,1} = \frac{1}{\sqrt{2}}\left(-\sqrt{2} - 2\sqrt{2}\right) = & -3 & c_{1,1} = \frac{1}{\sqrt{2}}(1-5) = & -2\sqrt{2} \\ & & c_{1,2} = \frac{1}{\sqrt{2}}(-3+4) = & \frac{1}{2}\sqrt{2} \\ & & c_{1,3} = \frac{1}{\sqrt{2}}(-3-4) = & -\frac{7}{2}\sqrt{2} \end{array}$$

$$\begin{array}{ll} c_{0,0} = \frac{1}{\sqrt{2}}\left(3\sqrt{2} + 0\right) & = 3 \\ c_{0,1} = \frac{1}{\sqrt{2}}\left(3\sqrt{2} - 0\right) & = 3 \\ c_{0,2} = \frac{1}{\sqrt{2}}\left(-2\sqrt{2} + (-2\sqrt{2})\right) & = -4 \\ c_{0,3} = \frac{1}{\sqrt{2}}\left(-2\sqrt{2} - (-2\sqrt{2})\right) & = 0 \\ c_{0,4} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\sqrt{2} + 0\right) & = \frac{1}{2} \\ c_{0,5} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}\sqrt{2} - 0\right) & = \frac{1}{2} \\ c_{0,6} = \frac{1}{\sqrt{2}}\left(-\frac{7}{2}\sqrt{2} + 0\right) & = -\frac{7}{2} \\ c_{0,7} = \frac{1}{\sqrt{2}}\left(-\frac{7}{2}\sqrt{2} - 0\right) & = -\frac{7}{2} \end{array}$$

We get the following reconstructed signal $f'_k = c_{o,k}$:

$$f' = (3, 3, -4, 0, \frac{1}{2}, \frac{1}{2} - \frac{7}{2}, -\frac{7}{2})$$

Problem 3.

- Dilation
  - First iteration with structuring element of 3 (m=1)

    $f \oplus B_3 = (..., (1), 3, 3, 3, 1, 2, 3, 3, 3, 2, 1, 2, 3, 4, 4, 4, 4, 4, 1, ...)$
  - Second iteration with structuring element of 3 (m=1)

    $(f \oplus B_3) \oplus B_3 = (..., (1), (3), 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 3, 4, 4, 4, 4, 4, 4, 4, (1), ...)$
  - Single iteration with structuring element of 5 (m=2)

    $$\begin{aligned} f \oplus B_5 &= (..., (1), (3), 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 3, 4, 4, 4, 4, 4, 4, 4, (1), ...) \\ &= (f \oplus B_3) \oplus B_3 \end{aligned}$$

One can see that the effect of the dilation is *additive* with respect to m.

- Erosion
  - First iteration with structuring element of 3 (m=1)

    $f \ominus B_3 = (..., 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 2, 3, 4, 1, 1, 1, ...)$
  - Second iteration with structuring element of 3 (m=1)

    $(f \ominus B_3) \ominus B_3 = (..., 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, ...)$
  - Single iteration with structuring element of 5 (m=2)

    $$\begin{aligned} f \ominus B_5 &= (..., 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, ...) \\ &= (f \ominus B_3) \ominus B_3 \end{aligned}$$

    Also in the case of the erosion one can see that the effect is *additive* with respect to m.

- Opening
  - First iteration with structuring element of 3 (m=1)

    $f \circ B_3 = (f \ominus B_3) \oplus B_3 = (..., 1, 1, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 3, 4, 4, 4, 1, 1, ...)$
  - Second iteration with structuring element of 3 (m=1)

    $$\begin{aligned} (f \circ B_3) \ominus B_3 &= (..., 1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1, 1, 2, 3, 4, 1, 1, 1, ...) \\ (f \circ B_3) \circ B_3 &= (..., 1, 1, 1, 1, 1, 2, 2, 2, 1, 1, 1, 2, 3, 4, 4, 4, 1, 1, ...) \\ &= f \circ B_3 \end{aligned}$$

    Since opening is a sieve operation, the first result is not changed by iterating it (structures that passed the first iteration, will also pass additional iterations).
  - Single iteration with structuring element of 5 (m=2)

    $f \circ B_5 = (f \ominus B_5) \oplus B_5 = (..., 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 1, 1, ...)$

    As to be expected from our previous consideration the effect of opening is *not additive*.

- Closing
  - First iteration with structuring element of 3 (m=1)

    $f \bullet B_3 = (f \oplus B_3) \ominus B_3 = (..., 1, 3, 1, 1, 1, 2, 3, 2, 1, 1, 1, 2, 3, 4, 4, 4, 1, 1, ...)$
  - Second iteration with structuring element of 3 (m=1)

    $(f \bullet B_3) \oplus B_3 = (..., (1), 3, 3, 3, 1, 2, 3, 3, 3, 2, 1, 2, 3, 4, 4, 4, 4, 4, 1, ...)$

$$
\begin{aligned}
(f \bullet B_3) \bullet B_3 &= (..., 1, 3, 1, 1, 1, 2, 3, 2, 1, 1, 1, 2, 3, 4, 4, 4, 1, 1, ...) \\
&= f \bullet B_3
\end{aligned}
$$

Since also closing is a sieve operation, the result is not changed either in this case by additional iterations.

– Single iteration with structuring element of 5 (m=2)

$$f \bullet B_5 = (f \oplus B_5) \ominus B_5 = (..., 1, 3, 3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 3, 4, 4, 4, 1, 1, ...)$$

As one can see also the effect of closing is *not additive*.

Problem 4.

$$
\begin{aligned}
S\left[f_p\right](u) &= \operatorname{stat}_x\left(f_p(x) - ux\right) \\
&= \left\{f_p(x) - ux | f_p'(x) - u = 0\right\} \\
&= \left\{cx^p - ux | pcx^{p-1} - u = 0\right\} \\
&= \left\{x\left(cx^{p-1} - u\right) | x^{p-1} = \frac{u}{pc}\right\} \\
&= \left\{x\left(\frac{u}{p} - u\right) | x = \operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}}\right\} \\
&= \left\{\operatorname{sgn}(u)\left|\frac{u}{pc}\right|^{\frac{1}{p-1}} \frac{u}{p}(1-p)\right\} \\
S\left[f_2\right](u) &= \left\{-\frac{u^2}{4c}\right\}
\end{aligned}
$$

So, for the case $p = 2$, the slope transform remains a quadratic function, i.e. paraboloids are preserved under the slope transform.