Image Processing and Computer Vision 2005/06
Example Solutions for Theoretical Assignments 2 (T2)

Problem 1.

(a) Let $k \in \{0, \ldots, M-1\}$ and $a \in \mathbb{R}$ with $a \neq 0$. Then, we get

$$\begin{aligned}
\widehat{f}_k(a) &= a \sum_{m=0}^{M-1} f_m \exp\left(-\frac{i2\pi km}{M}\right) \\
&= a\sqrt{M}\, \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(-\frac{i2\pi km}{M}\right) \\
&= a\sqrt{M}\, \widehat{f}_k\left(\frac{1}{\sqrt{M}}\right).
\end{aligned}$$

As one can see, the use of an arbitrary factor a results in a scaling of the Fourier coefficients by a factor of $a\sqrt{M}$ (compared to the transform for the orthonormal basis). If we apply the backtransform with a factor $b \in \mathbb{R}$ with $b \neq 0$ we then obtain

$$\begin{aligned}
f_k(a,b) &= b \sum_{m=0}^{M-1} \widehat{f}_m(a) \exp\left(\frac{i2\pi km}{M}\right) \\
&= b \sum_{m=0}^{M-1} a\sqrt{M}\, \widehat{f}_m\left(\frac{1}{\sqrt{M}}\right) \exp\left(\frac{i2\pi km}{M}\right) \\
&= ab\sqrt{M} \sum_{m=0}^{M-1} \widehat{f}_m\left(\frac{1}{\sqrt{M}}\right) \exp\left(\frac{i2\pi km}{M}\right) \\
&= abM\, \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \widehat{f}_m\left(\frac{1}{\sqrt{M}}\right) \exp\left(\frac{i2\pi km}{M}\right) \\
&= abM f_k\left(\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}\right) \\
&\overset{*}{=} abM f_k.
\end{aligned}$$

Evidently, the orginal signal is scaled once again. This time the scaling factor is given by abM. In order to obtain the original signal, a and b have to be chosen in such a way that $abM = 1$. This means that $b = \frac{1}{aM}$. For $a = \frac{1}{M}$ the corresponding factor of the backtransform is thus given by $b = 1$.

* It is known from the lecture that with the factors $a = b = \frac{1}{\sqrt{M}}$, the original signal is obtained, i.e. $f_k\left(\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}\right) = f_k$.

(b) Let $k \in \{0, \ldots, M-1\}$ be the indices of the Fourier transform of a signal of length $M \in \mathbb{N}$.

- Let M be even, i.e. $M = 2N$, where $N \in \mathbb{N}$.

  The contribution of the *lowest* frequency is then given by the coefficient at position $k = 0$. It represents the coefficient for the zero frequency (constant signal)

$$\begin{aligned} \exp\left(-\frac{i2\pi km}{M}\right) &= \cos\left(-\frac{i2\pi km}{M}\right) + i\sin\left(-\frac{i2\pi km}{M}\right) \\ &\overset{k=0}{=} \cos(0) + i\sin(0) \\ &= 1. \end{aligned}$$

  The contribution of the *highest* frequency is given by the coefficient at position $k = \frac{M}{2} = N$. This position corresponds to the most oscillatory frequency given by

$$\begin{aligned} \exp\left(-\frac{i2\pi km}{M}\right) &= \cos\left(-\frac{2\pi km}{M}\right) + i\sin\left(-\frac{2\pi km}{M}\right) \\ &\overset{k=M/2}{=} \cos\left(-\frac{2\pi m}{2}\right) + i\sin\left(-\frac{2\pi m}{2}\right) \\ &= \cos\left(\frac{2\pi m}{2}\right) - i\sin\left(\frac{2\pi m}{2}\right) \\ &= \cos(pim) - i\sin(\pi m) \\ &= (-1)^m - i\,0 \\ &= (-1)^m. \end{aligned}$$

  Please note: While the height of the corresponding frequency increases between $k = 0$ and $k = \frac{M}{2}$, it decreases between the indices $k = \frac{M}{2}$ and $k = M-1$. This can be explained as follows: Comparing the frequencies for the coefficients $k = a$ with $a \in \{1, \ldots, \frac{M}{2}\}$ and $k = M - a$, one gets

$$\begin{aligned}
\exp\left(-\frac{i2\pi km}{M}\right) &= \cos\left(-\frac{2\pi km}{M}\right) + i\sin\left(-\frac{2\pi km}{M}\right) \\
&\overset{k=a}{=} \cos\left(-\frac{2\pi am}{M}\right) + i\sin\left(-\frac{2\pi am}{M}\right) \\
&= \cos\left(2\pi - \frac{2\pi am}{M}\right) + i\sin\left(2\pi - \frac{2\pi am}{M}\right) \\
&= \cos\left(\frac{2\pi(M-a)m}{M}\right) + i\sin\left(\frac{2\pi(M-a)m}{M}\right) \\
&= \cos\left(-\frac{2\pi(M-a)m}{M}\right) - i\sin\left(-\frac{2\pi(M-a)m}{M}\right)
\end{aligned}$$

and

$$\begin{aligned}
\exp\left(-\frac{i2\pi km}{M}\right) &= cos\left(-\frac{2\pi km}{M}\right) + i\sin\left(-\frac{2\pi km}{M}\right) \\
&\overset{k=M-a}{=} \cos\left(-\frac{2\pi(M-a)m}{M}\right) + i\sin\left(-\frac{2\pi(M-a)m}{M}\right).
\end{aligned}$$

Obviously, apart from the different sign of the imaginary part, the coefficients a and $M - a$ represent the same frequency. This can also be seen from the example for $M = 8$ given in lecture 4, page 6.

- Let M be odd, i.e. $M = 2N + 1$, where $N \in \mathbb{N}$.

  The contribution of the *lowest* frequency is given once more by the coefficient at position $k = 0$ (see even case).

  The contribution of the *highest* frequency, however, is now given by *two* coefficients: $k = \frac{M-1}{2}$ and $k = \frac{M+1}{2}$. Since $\frac{M}{2} \notin \mathbb{N}_0$, the coefficient $k = \frac{M}{2}$ does not exist. In this case, the highest frequency is represented by the closest neighbouring coefficient (closest in terms of the index). Since all frequencies apart from the ones corresponding to $k = 0$ and $k = \frac{M}{2}$ (if existing) occur exactly twice (with positive and negative imaginary part), we have not one but two indices representing the highest frequency in the odd case.

Problem 2.

The Gaussian pyramid $\left\{v^N, \dots, v^0\right\}$ of a signal $u = (u_0, \dots, u_{2^N})^T$ is defined as

$$\begin{aligned} v^N &:= u, \\ v^{k-1} &:= R_k^{k-1} v^k \end{aligned}$$

for $k = N, \dots, 1$ with

$$R_k^{k-1} := \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & 0 & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

for $k \geq 2$ and $R_1^0 := \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$.

Applied to our signal $u := (32, 44, 26, 0, 8, 16, 31, 18, 0)^T$ we get

$$\begin{aligned} v^3 &= u \\ v^2 &= R_3^2 v^3 \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 32 \\ 44 \\ 26 \\ 0 \\ 8 \\ 16 \\ 31 \\ 18 \\ 0 \end{pmatrix} \\ &= (36, 24, 8, 24, 6)^T \\ v^1 &= R_2^1 v^2 \\ &= \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 36 \\ 24 \\ 8 \\ 24 \\ 6 \end{pmatrix} \\ &= (32, 16, 12)^T \\ v^0 &= R_1^0 v^1 \\ &= \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 32 \\ 16 \\ 12 \end{pmatrix} \\ &= 20 \end{aligned}$$

The Laplacian pyramid $\left\{w^N,\ldots,w^0\right\}$ of a signal $u=(u_0,\ldots,u_{2^N})^T$ with Gaussian pyramid $\left\{v^N,\ldots,v^0\right\}$ is defined as

$$\begin{array}{rcl} w^0 & := & v^0, \\ w^k & := & v^k - R^k_{k-1}v^k \end{array}$$

for $k=N,\ldots,1$ with

$$P_k^{k+1} \;\; := \;\; \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & \ldots & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 \end{pmatrix}$$

for $k \geq 1$ and $P_0^1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Applied to our signal, we get

$$\begin{aligned}
w^0 &= v^0 = 20 \\
w^1 &= v^1 - P_0^1 v^0 \\
&= \begin{pmatrix} 32 \\ 16 \\ 12 \end{pmatrix} - 20 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ -4 \\ -8 \end{pmatrix} \\
w^2 &= v^2 - P_1^2 v^1 \\
&= \begin{pmatrix} 36 \\ 24 \\ 8 \\ 24 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 32 \\ 16 \\ 12 \end{pmatrix} \\
&= (4, 0, -8, 10, -6)^T \\
w^3 &= v^3 - P_2^3 v^2 \\
&= \begin{pmatrix} 32 \\ 44 \\ 26 \\ 0 \\ 8 \\ 16 \\ 31 \\ 18 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 36 \\ 24 \\ 8 \\ 24 \\ 6 \end{pmatrix} \\
&= (-4, 14, 2, -16, 0, 0, 7, 3, -6)^T
\end{aligned}$$

Problem 3.

Let $C := (c_{ij}) := PR$.
First we show that all row sums of C are 1:

$$\begin{aligned}
\sum_j c_{ij} &= \sum_j \sum_k p_{ik} r_{kj} = \sum_j \sum_k p_{ik} \frac{p_{jk}}{\sum_l p_{lk}} \\
&= \sum_k p_{ik} \frac{\sum_j p_{jk}}{\sum_l p_{lk}} = \sum_k p_{ik} = 1.
\end{aligned}$$

Furthermore, because of

$$c_{ij} = \sum_k p_{ik} r_{kj} = \sum_k \frac{p_{ik} p_{jk}}{\sum_l p_{lk}} = \sum_k p_{jk} r_{ki} = c_{ji}$$

we know that C is symmetric.
Thus, all column sums of C are 1 as well, and we have

$$\sum_i y_i \;=\; \sum_i \sum_j c_{ij} x_j \;=\; \sum_j \Big(\sum_i c_{ij} \Big) x_j \;=\; \sum_j x_j .$$

This concludes the proof.

Problem 4.

(a) $f \;=\; (2,1,4,0)^T$

$\downarrow$ $\qquad\qquad\qquad$ $\searrow$

$\frac{3}{\sqrt{2}}, \frac{4}{\sqrt{2}}$ $\qquad\qquad\qquad$ $\frac{1}{\sqrt{2}}, \frac{4}{\sqrt{2}}$

$\downarrow$ $\qquad\qquad\qquad$ $\searrow$

$\frac{7}{2}$ $\qquad\qquad\qquad$ $\frac{-1}{2}$

wavelet transform:($\frac{7}{2}, \frac{-1}{2}, \frac{1}{\sqrt{2}}, \frac{4}{\sqrt{2}}$)

$f \;=\; (0,2,1,4)^T$
wavelet tranform:($\frac{7}{2}, \frac{-3}{2}, \frac{-2}{\sqrt{2}}, \frac{-3}{\sqrt{2}}$)

The wavelet transformation is not shift invariant, because there is a big difference between the two transformed signals. However, the coefficient that represents the average grey value is the same. Since one original signal was a shifted verstion of the other one this had to be expected.

(b) $f \;=\; (2,1,4,0)^T$

$$\hat{f}_0 = \frac{1}{2} \sum_{m=0}^{3} \exp(-\frac{i2\Pi 0m}{4})$$

$$= \frac{1}{2}(f_0 + f_1 + f_2 + f_3)$$

$$= \frac{1}{2}(2+1+4+0)$$

$$= \frac{7}{2}$$

$$\hat{f}_1 = \frac{1}{2}\sum_{m=0}^{3}\exp(-\frac{i2\Pi 1m}{4})$$

$$= \frac{1}{2}(f_0e^0 + f_1e^{\frac{-i\Pi}{2}} + f_2e^{-i\Pi} + f_3e^{\frac{-i3\Pi}{2}})$$

$$= \frac{1}{2}(2\cdot 1 + 1\cdot(-i) + 4\cdot(-1) + 0i)$$

$$= \frac{-2-i}{2}$$

$$\hat{f}_2 = \frac{1}{2}\sum_{m=0}^{3}\exp(-\frac{i2\Pi 2m}{4})$$

$$= \frac{1}{2}(f_0e^0 + f_1e^{-i\Pi} + f_2e^{-i2\Pi} + f_3e^{-i3\Pi})$$

$$= \frac{1}{2}(2\cdot 1 + 1\cdot(-1) + 4\cdot 1 + 0\cdot(-1))$$

$$= \frac{5}{2}$$

$$\hat{f}_3 = \bar{\hat{f}}_1 = \frac{-2+i}{2}$$

Fourier spectrum
$(0/\frac{7}{2})$
$(1/\frac{\sqrt{5}}{2})$
$(2/\frac{5}{2})$
$(3/\frac{\sqrt{5}}{2})$

Fourier transformation $f = (0,2,1,4)^T$

$$\hat{f}_0 = \frac{1}{2}\sum_{m=0}^{3}\exp(-\frac{i2\Pi 0m}{4})$$

$$= \frac{1}{2}(f_0 + f_1 + f_2 + f_3)$$
$$= \frac{1}{2}(0 + 2 + 1 + 4)$$
$$= \frac{7}{2}$$

$$\hat{f}_1 = \frac{1}{2}\sum_{m=0}^{3} \exp(-\frac{i2\Pi 1m}{4})$$
$$= \frac{1}{2}(f_0 e^0 + f_1 e^{\frac{-i\Pi}{2}} + f_2 e^{-i\Pi} + f_3 e^{\frac{-i3\Pi}{2}})$$
$$= \frac{1}{2}(0 \cdot 1 + 2 \cdot (-i) + 1 \cdot (-1) + 4i)$$
$$= \frac{-1 + 2i}{2}$$

$$\hat{f}_2 = \frac{1}{2}\sum_{m=0}^{3} \exp(-\frac{i2\Pi 2m}{4})$$
$$= \frac{1}{2}(f_0 e^0 + f_1 e^{-i\Pi} + f_2 e^{-i2\Pi} + f_3 e^{-i3\Pi})$$
$$= \frac{1}{2}(0 \cdot 1 + 2 \cdot (-1) + 1 \cdot 1 + 4 \cdot (-1))$$
$$= \frac{-5}{2}$$

$$\hat{f}_3 = \bar{\hat{f}}_1 = \frac{-1 - 2i}{2}$$

Fourier spectum
$(0/\frac{7}{2})$
$(1/\frac{\sqrt{5}}{2})$
$(2/\frac{5}{2})$
$(3/\frac{\sqrt{5}}{2})$

The Fourier spectrum is shift invariant.