Image Processing and Computer Vision 2005/06
Example Solutions for Theoretical Assignments 1 (T1)

Problem 1.

(a) We observe that with the given convolution kernel f one obtains for any discrete signal $g = (g_i)_{i\in\mathbb{Z}}$ the result

$$(g * f)_i = \sum_{j=-\infty}^{\infty} f_{i-j}\; g_j = \frac{1}{2}\, g_i + \frac{1}{2}\, g_{i-1}\ .$$

Applying this iteratively leads to

$$(f * f)_i = \begin{cases} 0, & i < 0 \text{ or } i > 2, \\ \frac{1}{4}, & i = 0, 2, \\ \frac{2}{4}, & i = 1 \end{cases}$$

$$(f * f * f)_i = \begin{cases} 0, & i < 0 \text{ or } i > 3, \\ \frac{1}{8}, & i = 0, 3, \\ \frac{3}{8}, & i = 1, 2 \end{cases}$$

$$(f * f * f * f)_i = \begin{cases} 0, & i < 0 \text{ or } i > 4, \\ \frac{1}{16}, & i = 0, 4, \\ \frac{4}{16}, & i = 1, 3, \\ \frac{6}{16}, & i = 2\ . \end{cases}$$

(b) For the given convolution kernel f and any continuous signal $g(x)$ we have

$$(g * f)(x) = \int_{-\infty}^{\infty} f(x - z)\; g(z)\, \mathrm{d}z \; = \int_{x-1}^{x+1} \frac{1}{2}\; g(z)\, \mathrm{d}z\ .$$

For $f * f$ we therefore obtain

$$
\begin{aligned}
(f * f)(x) &= \int\limits_{x-1}^{x+1} \frac{1}{2}\, f(z)\,\mathrm{d}z \\
&= \begin{cases} 0, & x \leq -2, \\ \int_{-1}^{x+1} \frac{1}{4}\,\mathrm{d}z, & -2 < x \leq 0, \\ \int_{x-1}^{1} \frac{1}{4}\,\mathrm{d}z, & 0 < x \leq 2, \\ 0, & x > 2 \end{cases}
\end{aligned}
$$

and thus

$$
(f * f)(x) = \begin{cases} 0, & x \leq -2, \\ \frac{2+x}{4}, & -2 < x \leq 0, \\ \frac{2-x}{4}, & 0 < x \leq 2, \\ 0, & x > 2\ . \end{cases}
$$

For $f * f * f$ we find

$$
\begin{aligned}
(f * f * f)(x) &= \int\limits_{x-1}^{x+1} \frac{1}{2}\,(f * f)(z)\,\mathrm{d}z \\
&= \begin{cases} 0, & x \leq -3, \\ \int_{-2}^{x+1} \frac{2+z}{8}\,\mathrm{d}z, & -3 < x \leq -1, \\ \int_{x-1}^{0} \frac{2+z}{8}\,\mathrm{d}z + \int\limits_{0}^{x+1} \frac{2-z}{8}\,\mathrm{d}z, & -1 < x \leq 1, \\ \int_{x-1}^{2} \frac{2-z}{8}\,\mathrm{d}z, & 1 < x \leq 3, \\ 0, & x > 3\ ; \end{cases}
\end{aligned}
$$

$$
(f * f * f)(x) = \begin{cases} 0, & x \leq -3, \\ \frac{x^2+6x+9}{16}, & -3 < x \leq -1, \\ \frac{6-2x^2}{16}, & -1 < x \leq 1, \\ \frac{x^2-6x+9}{16}, & 1 < x \leq 3, \\ 0, & x > 3\ . \end{cases}
$$

Finally, for $f * f * f * f$ one gets

$$(f * f * f * f)(x) = \int\limits_{x-1}^{x+1} \frac{1}{2}\,(f * f * f)(z)\,\mathrm{d}z$$

$$= \begin{cases} 0, & x \leq -4, \\ \int_{-3}^{x+1} \frac{z^2+6z+9}{32}\,\mathrm{d}z, & -4 < x \leq -2, \\ \int_{x-1}^{-1} \frac{z^2+6z+9}{32}\,\mathrm{d}z + \int_{-1}^{x+1} \frac{6-2z^2}{32}\,\mathrm{d}z, & -2 < x \leq 0, \\ \int_{x-1}^{1} \frac{6-2z^2}{32}\,\mathrm{d}z + \int_{1}^{x+1} \frac{z^2-6z+9}{32}\,\mathrm{d}z, & 0 < x \leq 2, \\ \int_{x-1}^{3} \frac{z^2-6z+9}{32}\,\mathrm{d}z, & 2 < x \leq 4, \\ 0, & x > 4\ ; \end{cases}$$

$$(f * f * f * f)(x) = \begin{cases} 0, & x \leq -4, \\ \frac{x^3+12x^2+48x+64}{96}, & -4 < x \leq -2, \\ \frac{-3x^3-12x^2+32}{96}, & -2 < x \leq 0, \\ \frac{3x^3-12x^2+32}{96}, & 0 < x \leq 2, \\ \frac{-x^3+12x^2-48x+64}{96}, & 2 < x \leq 4, \\ 0, & x > 4\ . \end{cases}$$

Problem 2.

In order to prove that the given set of vectors $\{v_0, ..., v_{M-1}\}$ form an orthonormal basis of the $\mathbb{C}^M$, we have to show the following three statements:

1) all vectors are from the $\mathbb{C}^M$, i.e. $v_i \in \mathbb{C}^M\ \forall\ i \in \{0, ..., M-1\}$ (already known)
2) the cardinality of the set is M, i.e. that we have M different vectors (also known)
3) the set of vectors is orthonormal with respect to $< \cdot, \cdot >$ and thus linearly independent (yet to show)

The latter statement is shown by proving the following:

For $p, q \in \{0, ..., M-1\}$ holds

$$< v_p, v_q >= \begin{cases} 1 & \text{if } p = q \\ 0 & \text{else} \end{cases}$$

(a) Let $p \in \{0, ..., M-1\}$, $q = p$.

Then we have:

$$
\begin{aligned}
< v_p, v_q > &= \sum_{m=0}^{M-1} \frac{1}{\sqrt{M}} \ \exp\left(\frac{i2\pi pm}{M}\right) \overline{\frac{1}{\sqrt{M}} \ \exp\left(\frac{i2\pi qm}{M}\right)} \\
&\overset{(M>0)}{=} \frac{1}{M} \sum_{m=0}^{M-1} \exp\left(\frac{i2\pi pm}{M}\right) \ \exp\left(-\frac{i2\pi qm}{M}\right) \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \exp\left(i2\pi \frac{(p-q)m}{M}\right) \\
&\overset{(p=q)}{=} \frac{1}{M} \sum_{m=0}^{M-1} \underbrace{\exp(0)}_{=1} \\
&= \frac{M}{M} \\
&= 1
\end{aligned}
$$

(b) Let $p, q \in \{0, ..., M-1\}$, $p \neq q$.

Then we have:

$$
\begin{aligned}
< v_p, v_q > &= \frac{1}{M} \sum_{m=0}^{M-1} \exp\left(\frac{i2\pi(p-q)m}{M}\right) \\
&= \frac{1}{M} \sum_{m=0}^{M-1} \left(\exp\left(i2\pi \frac{(p-q)}{M}\right)\right)^m \\
&\overset{(geom.sum)}{=} \frac{1}{M} \ \frac{1 - \left(\exp\left(i2\pi \frac{(p-q)}{M}\right)\right)^M}{1 - \exp\left(i2\pi \frac{(p-q)}{M}\right)} \\
&\overset{*}{=} \frac{1}{M} \ \frac{1 - \exp\left(i2\pi \frac{(p-q)M}{M}\right)}{1 - \exp\left(i2\pi \frac{(p-q)}{M}\right)} \\
&= \frac{1}{M} \ \frac{1 - \exp\left(i2\pi(p-q)\right)}{1 - \exp\left(i2\pi \frac{(p-q)}{M}\right)} \\
&\overset{**}{=} \frac{1}{M} \ \frac{1-1}{1 - \exp\left(\frac{i2\pi(p-q)}{M}\right)} \\
&= 0
\end{aligned}
$$

*Denominator:
Since $0 \leq p, q < M$ holds, it follows that $-M < p - q < M$ and therefore that $\frac{p-q}{M} \notin \mathbb{Z}$. Thus, $\exp\left(i2\pi\frac{(p-q)}{M}\right) \neq 1$. Evidently, the denominator cannot become zero.

**Nominator:
Since $(p - q) \in \mathbb{Z}$ holds, it follows that $\exp\left(i2\pi(p - q)\right)) = 1$. Thus, the nominator simplifies to zero.

Problem 3.

(a) Let $k \in \{1, \ldots, M - 1\}$. We get

$$\begin{aligned}
\widehat{f}_k &= \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(-\frac{i2\pi km}{M}\right) \\
&= \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \underbrace{\exp\left(\frac{i2\pi Mm}{M}\right)}_{=1} \exp\left(-\frac{i2\pi km}{M}\right) \\
&= \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(\frac{i2\pi(M-k)m}{M}\right) \\
&= \frac{1}{\sqrt{M}} \sum_{n=0}^{M-1} f_m \overline{\exp\left(-\frac{i2\pi(M-k)m}{M}\right)} \\
&\overset{f_m \in \mathbb{R}}{=} \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} \overline{f_m} \overline{\exp\left(-\frac{i2\pi(M-k)m}{M}\right)} \\
&= \overline{\widehat{f}_{M-k}}
\end{aligned}$$

We see that $\widehat{f}_k$ is just the complex conjugate of $\widehat{f}_{M-k}$ (note this only holds for real-valued signals).
Therefore, for a signal of even length $M = 2N$, we only need to know $\widehat{f}_0, \ldots, \widehat{f}_N$ to know all the fourier coefficients. $\widehat{f}_0$ and $\widehat{f}_N$ are always real numbers, all the other fourier coefficients may have nonvanishing imaginary part, i.e. we only need to know $Re\left(\widehat{f}_0\right), \ldots, Re\left(\widehat{f}_N\right)$ and $Im\left(\widehat{f}_1\right), \ldots, Im\left(\widehat{f}_{N-1}\right)$, which are exactly $2N = M$ real numbers. All the other fourier coefficients are redundant.
Likewise, for a signal of odd length $M = 2N + 1$, we only need to know $Re\left(\widehat{f}_0\right), \widehat{f}_1, \ldots, \widehat{f}_N$. All the other fourier coefficients are redundant.

(b) Let $m \in \{0, \dots, M-1\}, n \in \{0, \dots, N-1\}$. We see that

$$\begin{aligned}
\mathcal{F}\left[\widehat{f}\right](m,n) &= \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \widehat{f}(p,q) \exp\left(-i2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)\right) \\
&= \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \widehat{f}(p,q) \exp\left(i2\pi\left(-\frac{pm}{M} - \frac{qn}{N}\right)\right) \\
&= \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \widehat{f}(p,q) \exp\left(i2\pi\left(-\frac{pm}{M} - \frac{qn}{N}\right)\right) \\
&\quad \underbrace{\exp\left(i2\pi\left(\frac{pM}{M} + \frac{qN}{N}\right)\right)}_{=1} \\
&= \frac{1}{\sqrt{MN}} \sum_{p=0}^{M-1} \sum_{q=0}^{N-1} \widehat{f}(p,q) \exp\left(i2\pi\left(\frac{p(M-m)}{M} + \frac{q(N-n)}{N}\right)\right) \\
&= \mathcal{F}^{-1}\left[\widehat{f}\right](M-m, N-n) \\
&= f(M-m, N-n)
\end{aligned}$$

So the image is simply mirrored both in x and y-direction. Here, this is the same as rotating the image by π.

Problem 4.

The discrete Fourier transform is defined by

$$\hat{f}_p = \frac{1}{\sqrt{M}} \sum_{m=0}^{M-1} f_m \exp\left(-\frac{2\pi i p m}{M}\right) \qquad p \in \{0, ..., M-1\}$$

Since the input signal is real-valued, it suffices to compute the first $\lceil \frac{M}{2} \rceil$ frequencies (see Problem 3a)). For the computation we use Euler's formula

$$\exp(x) = \cos(x) + i \sin(x)$$

Direct application of the definition gives for the first signal $\left(-\frac{1}{2}, 0, \frac{1}{2}\right)$:

$$\begin{aligned}\hat{f}_0 &= \frac{1}{\sqrt{3}}\left(1f_0 + 1f_1 + 1f_2\right) \\ \hat{f}_1 &= \frac{1}{\sqrt{3}}\left(1f_0 + \frac{-1-i\sqrt{3}}{2}f_1 + \frac{-1+i\sqrt{3}}{2}f_2\right)\end{aligned}$$

and therefore

$$\begin{aligned}\hat{f}_0 &= 0 \\ \hat{f}_1 &= -\frac{\sqrt{3}}{4} + i\frac{1}{4} \\ \hat{f}_2 &= \overline{\hat{f}_1} = -\frac{\sqrt{3}}{4} - i\frac{1}{4}\end{aligned}$$

The Fouriers pectrum is given by

$$\begin{aligned}\left|\hat{f}_0\right| &= 0 \\ \left|\hat{f}_1\right| &= \sqrt{\left(-\frac{\sqrt{3}}{4}\right)^2 + \left(\frac{1}{4}\right)^2} = \frac{1}{2} \\ \left|\hat{f}_2\right| &= \sqrt{\left(-\frac{\sqrt{3}}{4}\right)^2 + \left(-\frac{1}{4}\right)^2} = \frac{1}{2}\end{aligned}$$

for the second signal $\left(\frac{1}{12}, -\frac{2}{3}, 0, \frac{2}{3}, -\frac{1}{12}\right)$:

$$\begin{aligned}\hat{f}_0 &= \frac{1}{\sqrt{5}}\left(1f_0 + 1f_1 + 1f_2 + 1f_3 + 1f_4\right) \\ \hat{f}_1 &= \frac{1}{\sqrt{5}}\left(1f_0 + e^{-\frac{2}{5}\pi}f_1 + e^{-\frac{4}{5}\pi}f_2 + e^{-\frac{6}{5}\pi}f_3 + e^{-\frac{8}{5}\pi}f_4\right) \\ \hat{f}_2 &= \frac{1}{\sqrt{5}}\left(1f_0 + e^{-\frac{4}{5}\pi}f_1 + e^{-\frac{8}{5}\pi}f_2 + e^{-\frac{12}{5}\pi}f_3 + e^{-\frac{16}{5}\pi}f_4\right)\end{aligned}$$

and therefore:

$$
\begin{aligned}
\hat{f}_0 &= 0 \\
\hat{f}_1 &\approx -0.30758192 + i0.423350189 \\
\hat{f}_2 &\approx 0.400751416 - i0.130212028 \\
\hat{f}_3 &= \overline{\hat{f}_2} \approx 0.400751416 + i0.130212028 \\
\hat{f}_4 &= \overline{\hat{f}_1} \approx -0.30758192 - i0.423350189
\end{aligned}
$$

The Fourier spectrum is given by

$$
\begin{aligned}
\left|\hat{f}_0\right| &= 0 \\
\left|\hat{f}_1\right| &\approx \sqrt{(-0.30758192)^2 + 0.423350189^2} = 0.27383202 \\
\left|\hat{f}_2\right| &\approx \sqrt{0.400751416^2 + (-0.130212028)^2} = 0.177556869 \\
\left|\hat{f}_3\right| &\approx \sqrt{0.400751416^2 + 0.130212028^2} = 0.177556869 \\
\left|\hat{f}_4\right| &\approx \sqrt{(-0.30758192)^2 + (-0.423350189)^2} = 0.27383202
\end{aligned}
$$