Lecture 9:
Computerised X-Ray Tomography I

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X-Ray Transmission Tomography

Motivation

- In conventional X-ray (projection) images many structures are superposed. This makes the interpretation difficult.
- In X-ray film images intensity differences of $2\,\%$ can be clearly identified. Air and bones can therefore easily be distinguished from muscle tissue. However, blood vessels and other soft-tissue details (e.g. in the heart) are not clearly visible.

Both problems can be alleviated by X-ray transmission tomography.

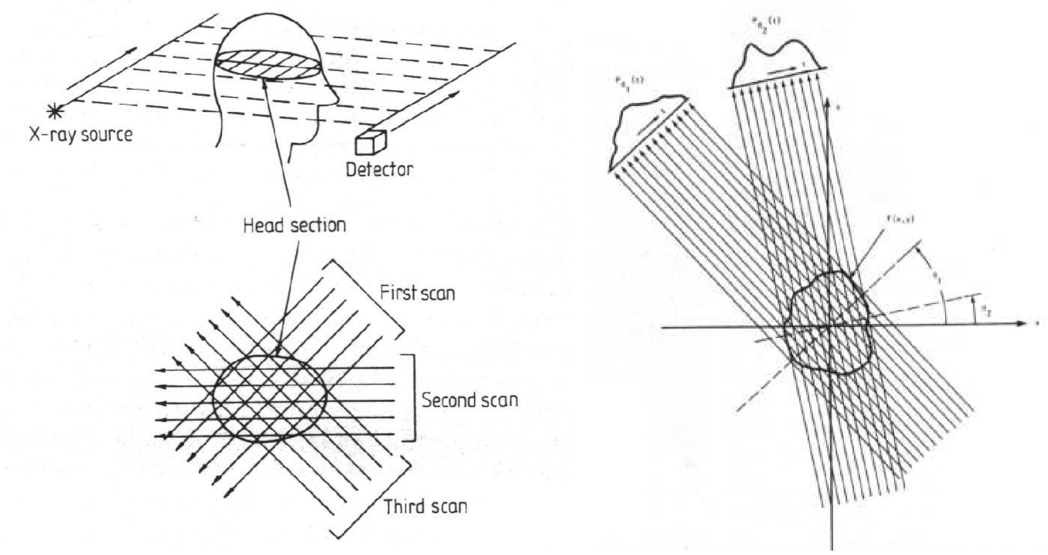
- three-dimensional imaging method
- reconstructs stack of two-dimensional sections of the object
- involves mathematical transformations of measured data (performed by a computer system within imaging device)
- called *computerised tomography*, *computer tomography (CT)* or *computer-aided tomography (CAT)*

X-Ray Transmission Tomography (2)

Idea of Tomography

We describe tomographic imaging of a 2-D section.

Take 1-D projection images of the section from different directions (angles). From these images together the 2-D section is to be reconstructed.

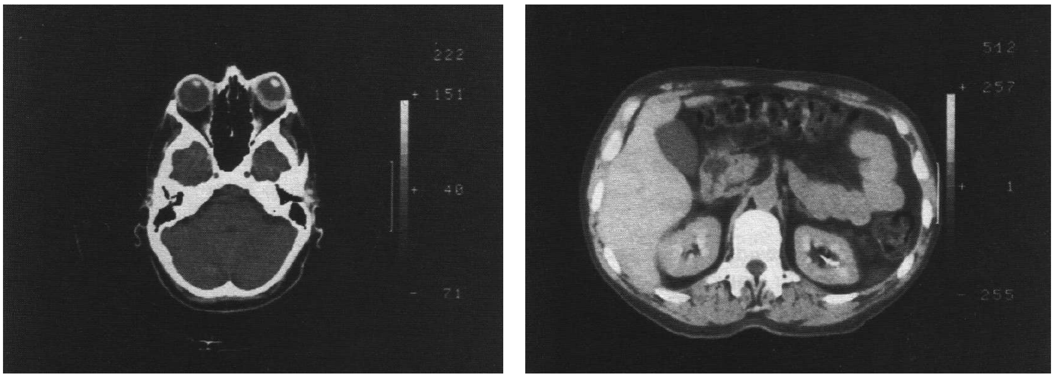


Schematic representations of X-ray section imaging. *(Left: Webb 1988; right: Kak, Slaney 2001)*

X-Ray Transmission Tomography (3)

Examples

The following two images give an example what can be achieved by this technique.



Sections of a human head on eye level (left) and the kidney region (right) obtained by computerised X-ray transmission tomography. *(Webb, 1988)*

X-Ray Transmission Tomography (4)

Historical Remarks

- J. Radon introduced in 1919 the mathematical transformation now known as *Radon transform*.
- S. I. Tetel'baum, B. I. Korenblyum, and A. A. Tyutin formulated in 1956–1958 the mathematical principles of tomographical reconstruction.
- A. M. Cormack further developed the mathematical theory.
- G. N. Hounsfield constructed the first working CT scanner in 1972. Cormack and Hounsfield shared the 1979 Nobel Prize for Medicine.

Left to right: Johann Radon (1887–1956), Allan McLeod Cormack (1924–1998), Godfrey Newbold Hounsfield (1919–2004). *(Images: http://www-groups.dcs.st-and.ac.uk/, http://nobelprize.org/)*

X-Ray Transmission Tomography (5)

Basic Modelling: Single Projection

- attenuation of a monochromatic X-ray beam follows the *Beer's Law* (Lectures 3 and 7).
- neglecting scatter and adapted to the situation of projecting a 2-D slice to 1-D gives

$$I(\xi) = C \exp\left(-\int \mu(\xi,\eta)\,\mathrm{d}\eta\right)$$

  for a projection along the η direction, with some constant C.
- total density along the line with fixed ξ:

$$\int \mu(\xi,\eta)\,\mathrm{d}\eta = -\ln\frac{I(\xi)}{C}\,.$$

- radiodensity μ (X-ray absorption density) measured in *Hounsfield units (HF)*

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Basic Modelling: Multiple Projections

- Up to now, (ξ, η) were orthogonal coordinates adapted to the particular projection.
- To consider multiple projections, these have to be embedded into a common coordinate frame, let it be (x, y).
- Then we have for the projection angle θ

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} .$$

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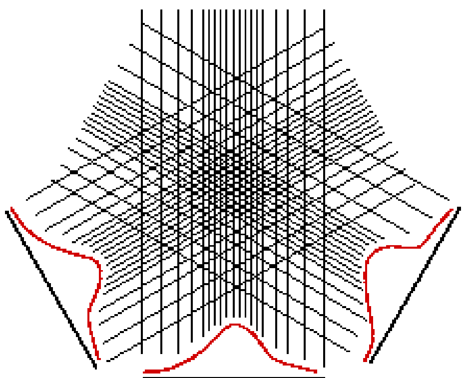
Simple Attempts to Tomographic Reconstruction

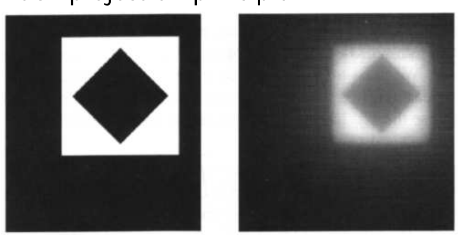
Naive Back-Projection

Basic Idea

- Project back each recorded 1-D projection into a "stripe image" in projection direction and superpose all these.
- Effectively, each pixel of the evolving section image receives the average of the projection values of all projection lines containing this pixel.
- The resulting image is rather blurred (but better than nothing).



Back-projection principle.



Original image and reconstruction by naive back-projection *(Epstein, 2003)*

Simple Attempts to Tomographic Reconstruction (2)

Why is the image obtained by back-projection blurred?

- Each single object point contributes to an entire bundle of projection lines which cross in that point.
- Back-projection smears the contribution of this point to all points on all projection lines.
- A single bright point on dark background is therefore smeared into an unsharp blob. *This blob has equal shape for all image points.* It is called *point-spread function (PSF)*.
- Each object point is blurred separately.
  The blurred image after back-projection is the superposition of all these blurred points.
  It is therefore the convolution of the sharp image with the PSF.

Simple Attempts to Tomographic Reconstruction (3)

Reconstruction by 2-D Deconvolution

Note that for the (unknown) sharp section image f, the naive back-projection image b and the convolution kernel h we have

$$b = f * h, \quad \Rightarrow \hat{b} = \hat{f}\hat{h} \, .$$

In principle, (an approximation of) the sharp section image could be obtained by *deconvolution techniques* (cf. Lecture 2). Note that in the Fourier domain this comes down to approximately computing

$$\hat{f} = \frac{\hat{b}}{\hat{h}}$$

(though this "inverse filtering" can usually not be used due to e.g. zeros of $\hat{h}$).

Often results are not favourable due to the effects of quantisation and discretisation. It is therefore not used in practice.

Deconvolution techniques are discussed in more detail in image processing lectures.

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The Radon Transform

In order to understand the tomographic reconstruction method used in computerised X-ray transmission tomography, we need a few mathematical prerequisites:

- Fourier transforms (cf. Lecture 2)
- Convolution (cf. Lecture 2)
- *Radon transform*

In the following, basic facts about the Radon transform are presented.
For more mathematical details, see e.g. Kak/Slaney, Epstein.

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The Radon Transform

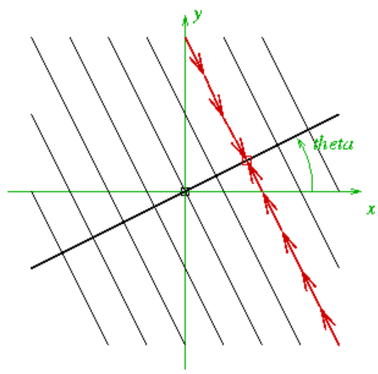
Assume we are given a function $f : \mathbb{R}^2 \to \mathbb{R}$, and an angle θ. The line l_θ through $(0,0)$ in direction θ (direction angle measured against the x axis) has the parameter representation

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = t \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} .$$

We want to *project* f perpendicularly onto this line. To this end, we integrate over perpendicular lines. In point $(x(t), y(t))$, the projection means to integrate over the line

$$\begin{pmatrix} x(t,s) \\ y(t,s) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + s \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} ,$$

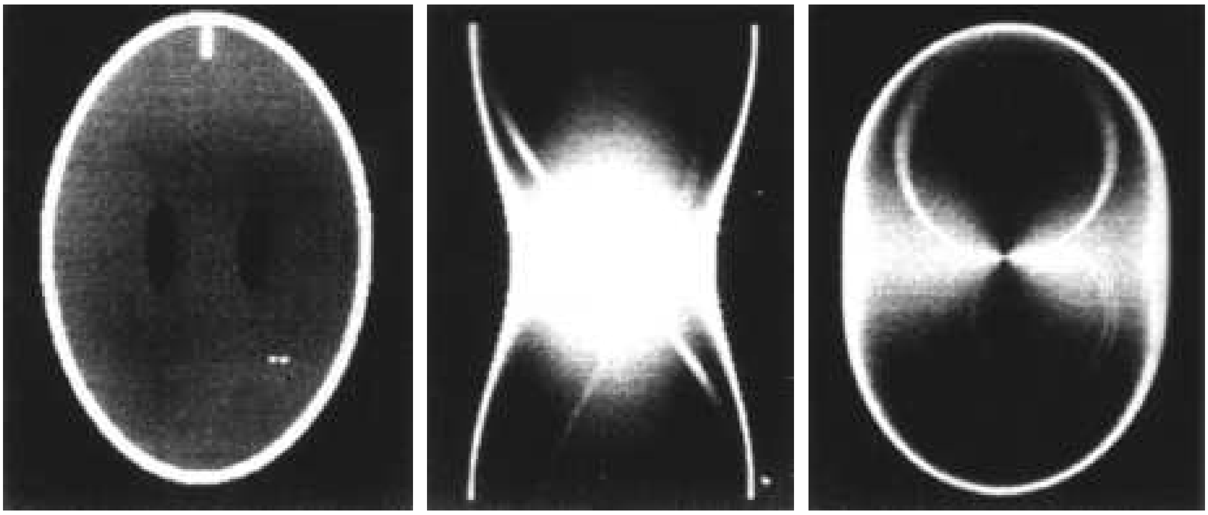
i.e. $$\int_{-\infty}^{\infty} f(x(t,s), y(t,s))\, \mathrm{d}s =: \mathcal{R}f(t,\theta)$$



which constitutes for each θ a function of t giving the projection values on the line l_θ, and as a whole, a function $\mathcal{R}f$ of t and θ.

The tomographic measurements give exactly the Radon transform $\mathcal{R}\mu$ of the radiodensity in the section plane.

The Radon Transform: Example



Left: Synthetic test image ("phantom"). **Middle:** Radon transform in rectangular coordinates: horizontal axis corresponds to angle, vertical to radius. **Right:** Radon transform in polar coordinates. *(Kak, Slaney 2001)*

Combining Radon and Fourier Transforms

Although the Radon transform is superficially a similar integral transform like the Fourier transform, it has not such a nice and simple inverse transform.

Consider, however, the 1-D Fourier transform of one projection, $r_\theta(t) := (\mathcal{R}f)(t,\theta)$:

$$\begin{aligned}
\hat{r}_\theta(w) &= \int_{-\infty}^{\infty} r_\theta(t)\mathrm{e}^{-2\pi \mathrm{i} t w}\,\mathrm{d}t = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta)\mathrm{e}^{-2\pi \mathrm{i} t w}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t\cos\theta - s\sin\theta, t\sin\theta + s\cos\theta) \\
&\qquad\qquad \mathrm{e}^{-2\pi \mathrm{i}((t\cos\theta - s\sin\theta)w\cos\theta + (t\sin\theta + s\cos\theta)w\sin\theta)}\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\mathrm{e}^{-2\pi \mathrm{i}(xw\cos\theta + yw\sin\theta)}\,\mathrm{d}y\,\mathrm{d}x \\
&= \hat{f}(w\cos\theta, w\sin\theta)
\end{aligned}$$

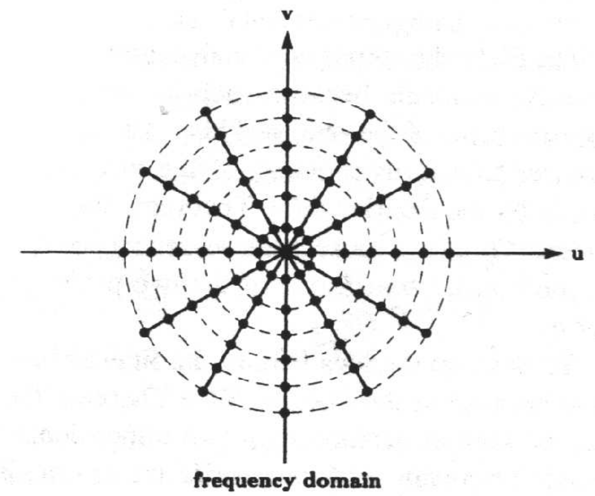
The Radon Transform (5)

Central Slice Theorem

The result of the previous slide can be summarised as follows:

The 1-D Fourier transform of the Radon transform of f on each line through the origin equals the 2-D Fourier transform of f along the same line.

In fact, we obtain the values of $\hat{f}$ for locations given by polar coordinates (w, θ) (see figure).



Recovery of the 2-D Fourier transform from Radon transform data. Each radial line contains the data from the 1-D Fourier transform of one projection. All together give the 2-D Fourier transform of the image to be reconstructed. *(Kak, Slaney 2001)*

Tomographic Reconstruction Using Integral Transforms (1)

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Tomographic Reconstruction Using Integral Transforms

Reconstruction via Fourier Transforms

The central slice theorem opens the way for the following reconstruction algorithm:

- Perform a 1-D Fourier transform of each measured linear projection.
- Assemble the 1-D Fourier transforms into one 2-D function $\hat{f}$ by placing each one into a line through the origin.
- Resample the 2-D function $\hat{f}$ to Cartesian coordinates.
- Perform a 2-D inverse Fourier transform of $\hat{f}$ to obtain the (sharp) reconstructed image $f = \mu$.

In practice, however, this algorithm is sensitive to perturbations in the data and numerically costly.

Reconstruction via Filtered Back-Projection

- Remember now the original idea of deconvolving the naive back-projection image.
- Using the link between Radon and Fourier transform established by the central slice theorem, the smearing of the back-projected image can be expressed as a smearing of the projected images:

$$\hat{b} = \hat{f} \cdot \hat{h}$$

$$\hat{b}(w \cos\theta, w \sin\theta) = \hat{r}_\theta(w) \cdot \hat{h}(w \cos\theta, w \sin\theta)$$

and by the rotational symmetry of h

$$\hat{b}(w \cos\theta, w \sin\theta) = \hat{r}_\theta(w) \cdot \hat{k}(w)$$

- If we could therefore back-project the inverse Fourier transform of $\frac{\hat{r}_\theta(w)}{\hat{k}(w)}$ instead of $r_\theta(w)$, we would obtain f instead of b.

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- We have therefore obtained a deconvolution problem in 1-D which is more convenient than the 2-D problem.
- One calculates the Fourier back-transform $p_\theta(t)$ of

$$\hat{p}_\theta(w) := \hat{r}_\theta(w) \cdot \hat{q}(w) \approx \frac{\hat{r}_\theta(w)}{\hat{k}(w)}$$

where q is an *approximative convolution inverse* of k, i.e. a convolution function (*"convolution kernel"*) which approximately reverses the smearing by k.

- The kernel q can be precomputed, and the calculation of p can be done via the Fourier domain or even directly.
- Feeding the prefiltered projections $p_\theta(t)$ instead of $r_\theta(t)$ into the back-projection procedure yields directly a sharp reconstruction image u which approximates f.
- This is the most common way to implement tomographic reconstruction.
- By varying q the properties of the reconstructed image can be tuned, e.g. for detail sharpness, for contrast of contours, for reduction of certain artifacts.

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Algebraic Reconstruction Techniques

- There exist alternative approaches to the tomographic reconstruction problem. In the context of X-ray transmission tomography, these don't play an important role nowadays but they have been important in the history. Moreover, they are still in use in the context of other types of tomographic methods.
- *Algebraic Reconstruction Techniques (ART)* basically rely on discrete formulations of the reconstruction problem.
- Incidences between pixels in the 2-D section that is to be reconstructed and pixels in the 1-D projections are modelled into large systems of equations, linear or nonlinear.
- These systems of equations are typically solved by iterative methods.

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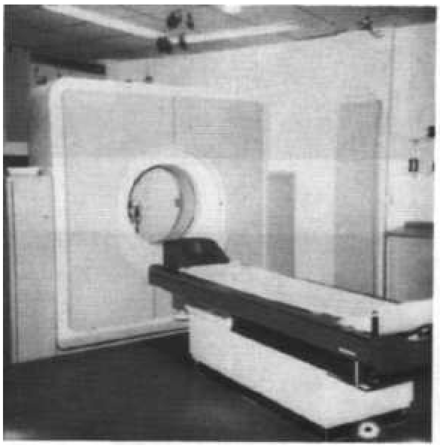
Computer Tomography Scanners

- The purpose of a CT scanner is to record the projected X-ray images of an object from different directions.
- Scanners differ in the details of their design, with consequences for scan times and for the exact geometry of acquired projections.

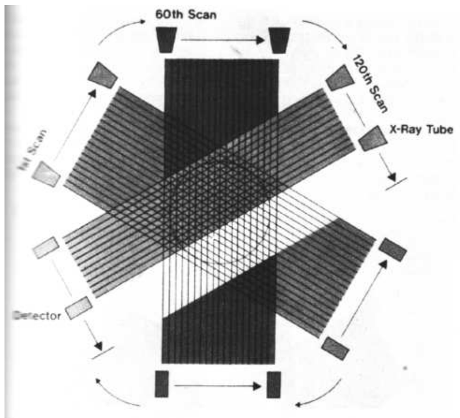


A typical medical CT scanner. The housing with circular opening is called *gantry*. The patient moves through the gantry while subsequent cross-sections are imaged.

CT Scanners (2)

First Generation Scanners

- One collimated (bundled) X-ray beam from a single source sent through the object to one collimating detector is used to measure transmission along one line.
- For a full 1-D projection, source and detector are moved in translation across the object.
- After rotation by a small angular step, the same procedure is repeated to obtain the next projection etc.



First generation CT scanner, schematic view. *(Maravilla and Pastel, 1978; reproduced from Webb, 1988)*

- Calibration is easy since the same source and detector are used for all measured values.
- High image quality is possible: the contribution of diffracted X-rays is minimal.
- Image acquisition time is long: typically several minutes for a single cross-section.

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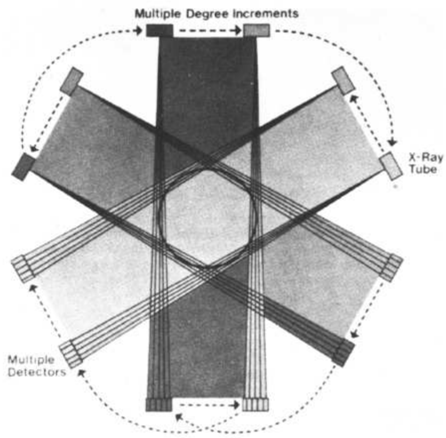
CT Scanners (3)

Second Generation Scanners

- Multiple detectors (a 1-D array) are used together with a fan of beams emitted by a single source (opening angle of the fan: typically $\approx 10°$).
- Source and detector array are moved across the object to scan an entire projection.
- In one scan, projections for a set of different angular directions are measured simultaneously: one for each detector in the array.



Second generation CT scanner, schematic view. *(Maravilla and Pastel, 1978; reproduced from Webb, 1988)*

- Consequentially, the angular step between subsequent linear scans is larger; it corresponds to the opening angle of the X-ray fan.
- Calibration should ensure that the measurements of different detectors are comparable. However, each projection is measured by a single detector.
- One section is measured in approx. $20\,\mathrm{s}$. By holding breath, inner organs can be imaged without motion artifacts.

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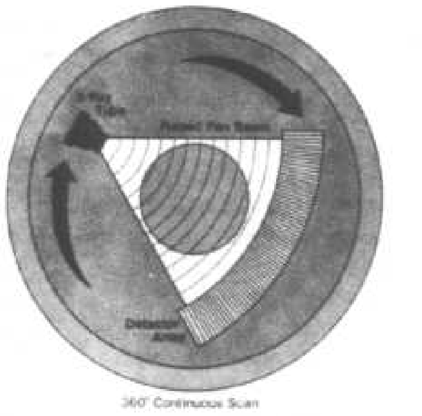
Third Generation Scanners

- A larger fan beam is used that covers the entire object cross-section.
- No linear scan is needed, instead source and detector array rotate continuously around the gantry while the X-ray source emits pulses of radiation.
- Calibration is critical since each projection contains data measured by different detectors.



Third generation CT scanner, schematic view. *(Maravilla and Pastel, 1978; reproduced from Webb, 1988)*

- One sectional image is measured in about $4 \dots 5\,\mathrm{s}$.

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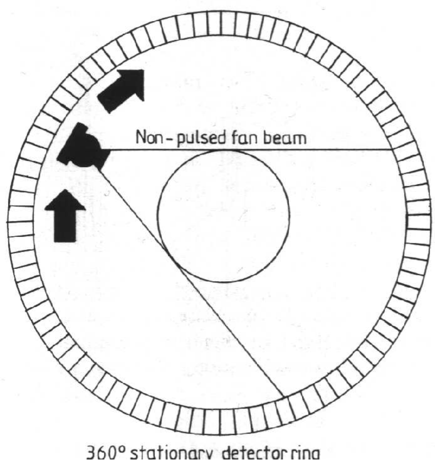
Fourth Generation Scanners

- A stationary ring of detectors (typically ≈ 1000) is used.
- Only the X-ray source rotates around the gantry opening, emitting a non-pulsed X-ray fan.
- Calibration is easier since normally each detector is hit at some time during imaging by an unattenuated beam (a beam that misses the object).
- Scanning speed is comparable to third-generation scanners.



Fourth generation CT scanner, schematic view. *(Maravilla and Pastel, 1978; reproduced from Webb, 1988)*

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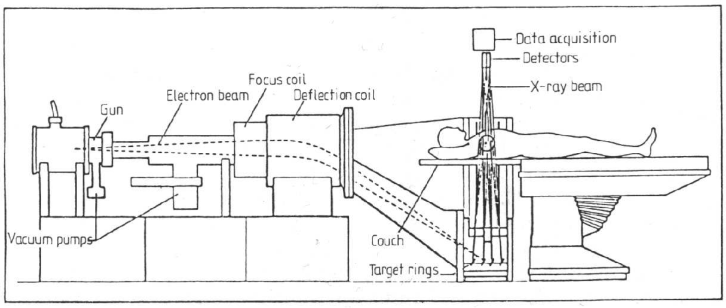
Fifth Generation Scanners



Fifth generation CT scanner, schematic view. *(Image by Imatron; reproduced from Webb, 1988)*

- No mechanically moving parts remain in the gantry.
- A detector ring is used.
- A large, specially shaped X-ray tube with target in the shape of a circular arc of approx. $210°$ is used. The effective X-ray source is moved around the object by redirecting an electron beam to different positions on this target.
- Image acquisition times around $0.1\,\mathrm{s}$ can be achieved.

Extensions

The mathematical techniques to reconstruct 2-D sections can be adapted in several ways to different scanner geometries, to avoid as much as possible re-sampling of data during calculations.

- Particularly, reconstruction algorithms can be formulated directly for fan-beam projections instead of parallel projection images.

More advanced mathematical reconstruction techniques allow to abandon the restriction to separately recorded 2-D sections. Extensions go in two directions:

- **Spiral or Helical Scanners.** This type of scanners is still close to a fourth- or fifth-generation scanner. However, the object is in a continuous translatory motion while the X-ray source rotates around the gantry opening. Consequently, the X-ray source performs a spiral, a.k.a. helical movement relative to the object.
- **Cone-Beam Scanning.** Instead of using a fan beam and a 1-D detector array, an entire cone of X-ray is sent from the point-shaped source to a 2-D detector array. Beams are no longer perpendicular to a common axis. The reconstruction now involves full 3-D transformations.

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Summary

- Computerised tomography (CT) is an X ray imaging technique that reconstructs 3D absorption densities.
- To this end one needs projections from many directions.
- Naive backpropagation gives blurred results.
  Direct 2D deblurring is expensive and sensitive to discretisation and quantisation.
- Tomographic measurements compute the Radon transform.
- The Central Slice Theorem allows to express the 1D Fourier transform of the Radon transform by a 2D Fourier transform.
- This allows reconstruction by resampling in the Fourier domain.
- The most popular reconstruction is filtered backprojection that performs 1D deconvolution before backprojection.
- Numerous scanning geometries exist that allow fast reconstructions in good quality.

References

- S. Webb, *The Physics of Medical Imaging.* Institute of Physics Publishing, Bristol 1988.
- A. C. Kak, M. Slaney, *Principles of Computerized Tomographic Imaging.* SIAM, Philadephia 2001.
- C. L. Epstein, *Introduction to the Mathematics of Medical Imaging.* Pearson, Upper Saddle River 2003.

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