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# Lecture 2:
# Basic Concepts II

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# Mathematical Tools

## Introductory Remark

The concepts of *Fourier transform* and *convolution* are essential for understanding the sampling process, and will also be used in different contexts later in the lecture.

Within this lecture, our coverage of these topics remains necessarily superficial, and mathematically not strict. Some hints where to find the rest of the story:

- Analysis textbooks give a full mathematical description.
- The Fourier transforms and convolution (which we consider here) are presented in more detail in the *Image Processing and Computer Vision* course.
- Also signal processing textbooks inform about Fourier transforms and convolution. The books by Kak/Slaney and Epstein in the references of this lecture also contain sections on Fourier transforms and related material.

## Fourier Transform in 1D

Given a 1D function $f : \mathbb{R} \to \mathbb{R}$, its *Fourier transform* is defined as the function

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi \mathrm{i} \xi x} \, \mathrm{d}x$$

where calculations are done in complex numbers. Remember $\mathrm{i}^2 = -1$.

The complex-valued function $\hat{f}(\xi)$ describes $f$ as superposition of harmonic 1D waves (scaled and translated sine functions) of different frequencies and phases.
The variable $\xi$ gives the frequency, while the phase is encoded in the phase of the complex value $\hat{f}(\xi)$.

**Remarks**

1. The Fourier transform $\hat{f}$ of a real-valued function $f$ is symmetric w.r.t. the origin: $\hat{f}(-\xi) = \overline{\hat{f}(\xi)}$ (bar denoting complex conjugate)
2. The high frequency components are particularly important to represent fine details.

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## Inverse Fourier Transform in 1D

From $\hat{f}$, the function $f$ can be recovered via the *inverse Fourier transform* (Fourier back-transform)

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) \mathrm{e}^{2\pi \mathrm{i} \xi x} \, \mathrm{d}\xi.$$

Note that the spatial variable $x$ and the frequency $\xi$ have swapped, and the sign of the exponent has changed.

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## The Fourier Transform in 2D

Given a 2D function $f : \mathbb{R}^2 \to \mathbb{R}$, its Fourier transform is defined as

$$\hat{f}(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \mathrm{e}^{-2\pi \mathrm{i}(\xi x + \eta y)} \, \mathrm{d}x \, \mathrm{d}y \; .$$

$\hat{f}$ describes a decomposition into harmonic waves (planar sinusoidal waves in 2D). The vector $(\xi, \eta)^{\mathrm{T}}$ gives the 2D frequency, i.e. a unit vector in direction of $(\xi, \eta)^{\mathrm{T}}$ describes the propagation direction, and the length $\left|(\xi, \eta)^{\mathrm{T}}\right|$ the frequency.

The inverse transform reads

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\xi, \eta) \mathrm{e}^{2\pi \mathrm{i}(\xi x + \eta y)} \, \mathrm{d}\eta \, \mathrm{d}\xi \; .$$

*Separability:* The 2D transform can be performed by successively doing 1D Fourier transforms w.r.t. $x$ alone and $y$ alone:

$$f(x, y) \to \tilde{f}(\xi, y) \to \hat{f}(\xi, \eta) \; .$$



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## Fourier Transform Example

**Left:** Test image, $256 \times 256$ pixels. **Right:** Fourier spectrum (i.e. magnitude of complex-valued Fourier transform) computed by Fast Fourier transform (FFT). Grey-values represent logarithmically rescaled spectrum entries. The discrete Fourier spectrum corresponds to the region of the full continuous-valued Fourier spectrum up to the Nyquist frequency, with zero in the middle, if the image is assumed to be band-limited (compare Slide 2:14).

# Mathematical Tools (6)



## Convolution in 1D

- Convolution can be understood as an operation of smearing a signal or image represented by a function.
- Assume further we have a function $h : \mathbb{R} \to \mathbb{R}$ which represents the smearing.
- The *convolution* of $f$ with $h$ is defined as

$$(f * h)(x) = \int_{-\infty}^{\infty} f(x - x')\, h(x')\, \mathrm{d}x' \ .$$

# Mathematical Tools (7)

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## Convolution in 2D

For functions of two variables $f : \mathbb{R}^2 \to \mathbb{R}$, $h : \mathbb{R}^2 \to \mathbb{R}$ we have

$$(f * h)(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - x', y - y')\, h(x', y')\, \mathrm{d}x'\, \mathrm{d}y' \ .$$

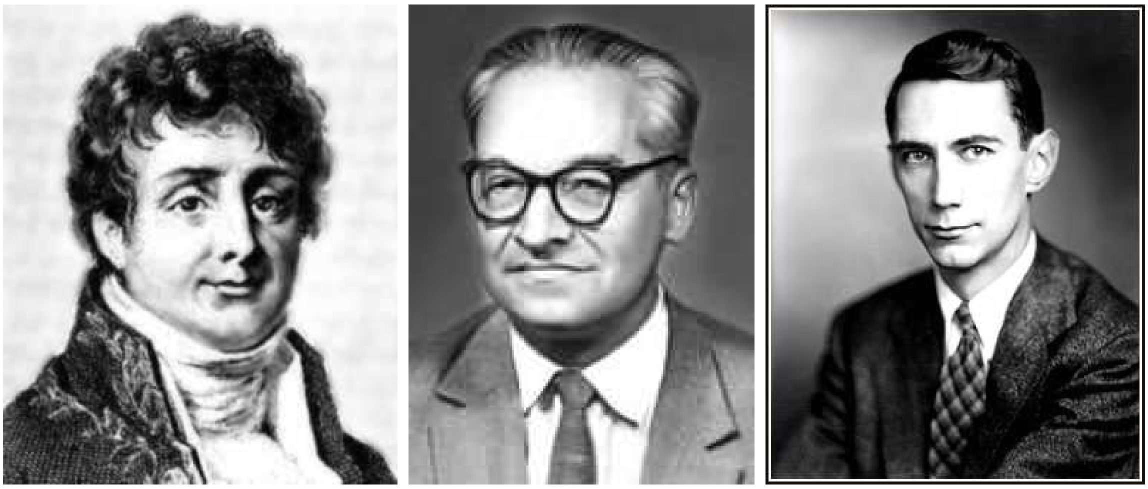
## Convolution Theorem

Convolution and Fourier transform are related via

$$\widehat{(f * h)}(\xi) = \hat{f}(\xi) \cdot \hat{h}(\xi) \qquad \text{over } \mathbb{R} \ ,$$

$$\widehat{(f * h)}(\xi, \eta) = \hat{f}(\xi, \eta) \cdot \hat{h}(\xi, \eta) \qquad \text{over } \mathbb{R}^2 \ .$$

## History



**Left:** Jean Baptiste Joseph Fourier (1768–1830), French mathematician. Introduced harmonic analysis. *(Image: http://www-history.mcs.st-andrews.ac.uk/)*. **Middle:** Vladimir Aleksandrovich Kotel'nikov (1908–2005), Russian mathematician. *(Image: public domain, copied from Wikipedia)*. **Right:** Claude Elwood Shannon (1916–2001). *(Image: public domain, copied from Wikipedia)*. **Missing:** Harry Nyquist (1889–1976), Swedish-American mathematician. Kotel'nikov, Shannon and Nyquist are the discoverers of the Sampling Theorem.



## Sampling in 1D (1)

# Sampling in 1D

- Study now sampling process, i.e. reduction of (object) function $f$ on continuous domain $\Omega_{\text{phys}}$ to (image) function $g$ on discrete domain $\Omega$.
- To start with, consider only the 1D case, i.e. $\Omega_{\text{phys}} = \mathbb{R}$.
- Assume that $\Omega$ is a regular grid, $\Omega = \{kh_s \mid k \in \mathbb{Z}\}$, with a fixed grid size $h_s > 0$.

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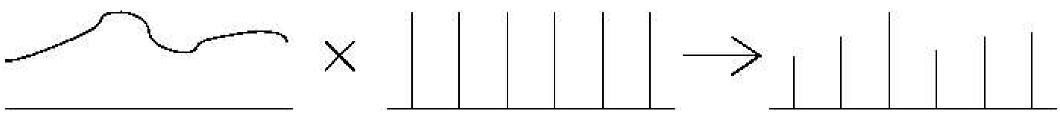
## Sampling in 1D (2)

### Ideal Sampling

- Sample $f$ at each point of $\Omega$, i.e.

$$g(x) = f(x)\ , \qquad x \in \Omega\ .$$

- Physically possible only in rare conditions (e.g. meteorological measurements), we discuss this later.
- In $\mathbb{R}$, sampling means to *multiply* $f$ by a comb of sharp peaks of height $1$, a so-called *delta comb*. It consists of peaks equally spaced at the sampling interval length $h_\mathrm{s}$.

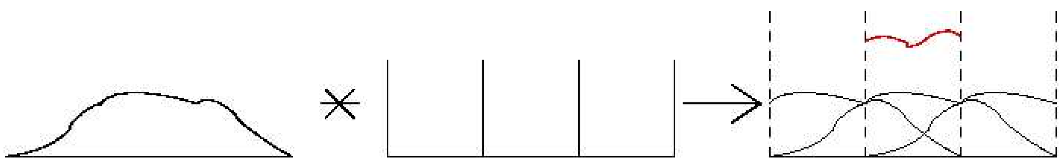




## Sampling in 1D (3)

### Aliasing

- The Fourier transform of a delta-comb is also a delta-comb, equally spaced at the sampling frequency $1/h_\mathrm{s}$
- Thus, the Fourier transform $\hat{f}$ is *convolved* with a delta-comb by the sampling.
- Thus, $\hat{g}$ is $1/h_\mathrm{s}$-periodic.
  It is sufficient to consider an interval of length $1/h_\mathrm{s}$.
  By default one uses $[-\frac{1}{2h_\mathrm{s}}, \frac{1}{2h_\mathrm{s}})$.
  The frequency $1/2h_\mathrm{s}$ is called *Nyquist frequency*.
- All frequency components of $\hat{f}$ are aggregated in this interval.
  This wrapping of high frequencies into lower frequencies is called *aliasing*.



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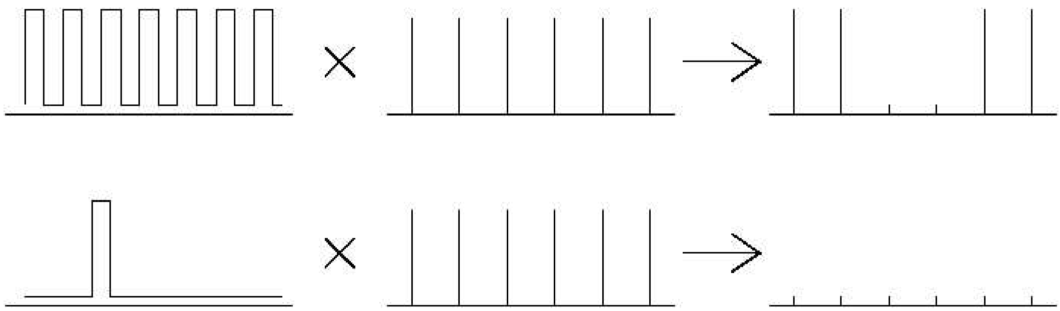
## Sampling in 1D (4)

- Obviously, aliasing constitutes a loss of information.
- Even worse, it leads to misrepresentation of information.
- Examples:
  - Periodic structures whose frequency is higher than the Nyquist frequency appear as low-frequency structures.
  - Fine details between sampling points can be lost.





## Sampling in 1D (5)

**Sampling Theorem**

- Aliasing is no problem if $\hat{g}$ is just the restriction of $\hat{f}$ to the interval $[-1/2h_\mathrm{s}, 1/2h_\mathrm{s})$.
- This can be guaranteed if $\hat{f}$ contains only frequencies below $1/2h_\mathrm{s}$, i.e. $\hat{f}(\xi) = 0$ whenever $|\xi| \geq 1/2h_\mathrm{s}$.
- A function $f$ with this property is called *band-limited*.
- **Theorem.** *A function $f$ that is band-limited by the Nyquist frequency can be uniquely reconstructed from the sampled function $g$.*

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# Sampling in 1D (5)

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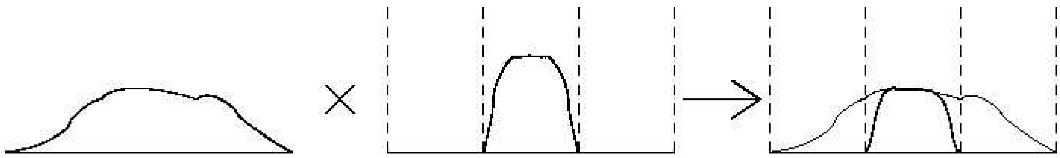
## Low-Pass Filtering

- Real-world object functions are often not band-limited (at least not with limiting frequencies achievable for sampling)
- In order to avoid misrepresentation and data loss by aliasing, one is interested in making the function band-limited before reducing it to the discrete grid.
- This can be achieved by low-pass filtering.
- (Ideal) low-pass filtering consists in convolution with a function $h$ whose Fourier transform $\hat{h}$ has its support in the desired interval, e.g. $(-1/2h_\mathrm{s}, 1/2h_\mathrm{s})$.



# Sampling in 1D (6)

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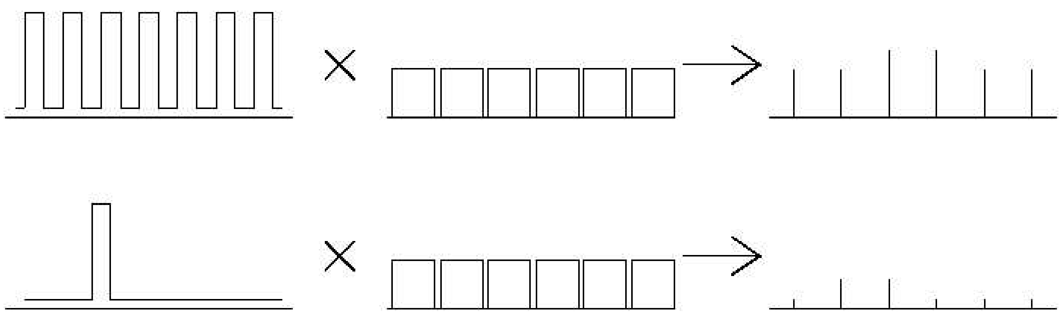
## Point-Spread Function

- In practically relevant cases, the measurement process (optics, sensors, etc.) already introduce some convolution kernel $h$ that is approximately a low-pass filter. The entire sampling process then reads

$$g(x) = (f * h)(x) \ , \qquad x \in \Omega \ .$$

- $h$ is called *point-spread function (PSF)* or *point-source response function (PSRF)*.
- **Example:** $h$ is box-function of length $h_\mathrm{s}$, i.e. the sensor integrates the object function over the sampling interval. Not a good low-pass, may create artifacts:



### Modulation Transfer Function

- The full, complex-valued Fourier transform $\hat{h}$ is called *optical transfer function (OTF)* in optical systems.
- The magnitude $|\hat{h}|$ is called *modulation transfer function (MTF)*.
  The MTF does not contain the full information of $h$ but often only this part is accessible for measurement.
- The phase part of $\hat{h}$ is called *phase transfer function*.



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## Sampling in 2D

### Regular Grids

- Assume $f : \mathbb{R}^2 \to \mathbb{R}$ is a 2D object function which is sampled on a regular grid $\Omega = \{(kh_x, lh_y) \mid k, l \in \mathbb{Z}\}$.
- By using the 2D Fourier transform and 2D convolutions, the sampling process can be analysed almost analogously to the 1D case.
- The concepts of PSF and MTF are analogous to the 1D case.

However, this scenario is not very realistic.

## Realistic PSF

- To match the rectangular sampling in sensors, the convolution kernels should be separable:
$$h(x, y) = h_1(x) \cdot h_2(y).$$
- Physical effects (e.g. optics), however, may create rotationally symmetric PSF, e.g.
$$h(x, y) = h_0(\sqrt{x^2 + y^2}).$$
More difficult to analyse mathematically (aliasing, sampling theorem), and less sharp estimates.
Sometimes $h_0$ itself will be denoted as PSF, and also the MTF is sometimes given just as a radial function.
- Most realistic model includes separable components due to sensor design with rotationally symmetric components due to physics:
$$h(x, y) = (h_1(x) \cdot h_2(y)) * [h_0(\sqrt{x^2 + y^2})](x, y).$$



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## Deconvolution Problem

- Due to physical limitations of the imaging system, or degradations already present in the measured object function, the total PSF in an image may be "wider" than desirable by the sampling theorem.
- may lead to a *deconvolution problem*:
*Assume we are given an image $g$ which is degraded by convolution with a certain kernel $h$, i.e.*
$$g = h * \tilde{g}$$
*with an unknown sharper image $\tilde{g}$, or with some noise $n$,*
$$g = h * \tilde{g} + n \ .$$
*Recover (a good approximation to) the sharper image $\tilde{g}$.*

## Sampling in 2D (4)

- Deconvolution is an image *processing* problem.
  Thus, it will not be covered in this lecture.
  Some image acquisition methods, however, lead to similar problems.

- When deconvolving sampled data, one should distinguish two effects:
  – contributions to the PSF before discretisation,
  – contributions during the discretisation.
  Do not deconvolve the latter: You can't cheat the sampling theorem.

- Deconvolution methods are among the research topics in our group
  (bachelor/master thesis topics regularly available)



## Sampling in 2D (5)

### Example for a Deconvolution Problem

**Left:** Paris at dusk from Eiffel tower, blurred by camera movement. **Right:** Restoration by a deconvolution method. *(Welk, Theis, Weickert 2005)*

# Sensor Fusion

## Definition

- To obtain a more comprehensive information on an object, it can be useful to combine information from more than one imaging sensor.
- Sources can be homogeneous (same type) or heterogeneous (different types). Further parameters like resolution can be equal or different.
- *Sensor fusion* joins data from separate sources to a single data set. It contains more information than any single data source.
- We give a few examples. The particular image acquisition methods are topic of later lectures.



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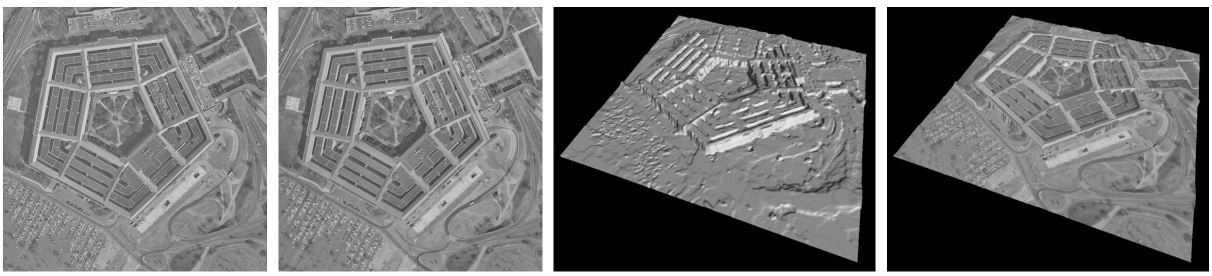
## Examples

- **Stereo Imaging** (equal sensor type, equal resolution):

  Given two photographs of the same object, taken from different viewpoints, synthesise 3D information.

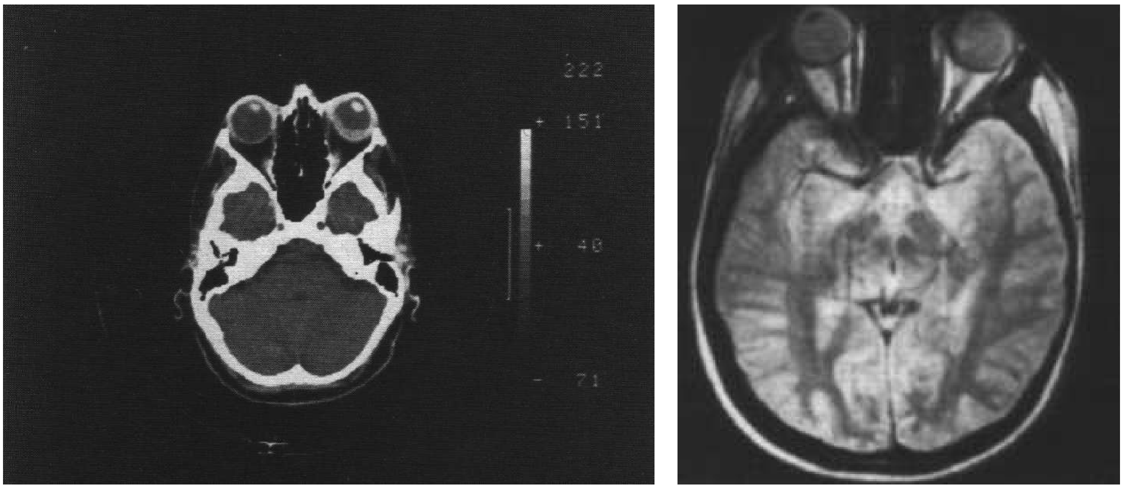


**Left two images:** *Pentagon* stereo image pair. *(http://vasc.ri.cmu.edu/idb/html/stereo/ ).* **Right two images:** Stereo reconstruction without and with texture. *(Slesareva et al. 2005)*

## Sensor Fusion (3)

- **CAT and MRI** (different sensor types, resolution comparable):

  Computerised X-ray tomography (CAT) and magnetic resonance imaging (MRI) yield 3D images of the human body.
  Each of them has strengths and weaknesses for certain anatomical structures.
  Fusing them gives more complete information.




Sections of human heads on eye level obtained by computerised X-ray tomography (left) and MRI (right). *(Both images: Webb, 1988)*

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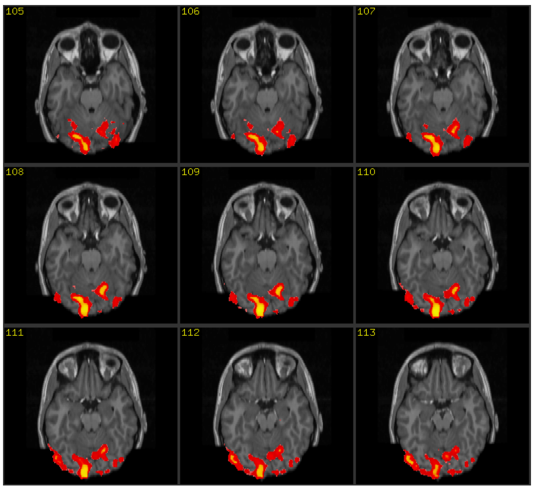
## Sensor Fusion (4)

- **MRI and fMRI** (different sensor types, different resolution):

  Standard magnetic resonance imaging (MRI) gives static images of high resolution.
  Functional MRI (fMRI) gives dynamic images of low resolution (brain activity).
  Data fusion helps gaining insights into cognitive processes.




Fusion of MRI and fMRI. fMRI information is displayed in pseudocolours in a higher resolution anatomical MRI scan (grey-values). *(S. Smith 1998, http://www.fmrib.ox.ac.uk/fmri_intro)*

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## Sensor Fusion (5)

### Correspondence Problems

- Each instance of sensor fusion gives rise to a *correspondence problem*, i.e. the task to find which data in different images belong to the same object points.
  - In stereo imaging, this is known as *stereo problem*.
  - In the medical context, one speaks of *image registration*.
- Even for equal resolution, registration between different sensor types is nontrivial (e.g. due to slight movements between different measurements).
- As correspondence problems, too, are image processing problems, we do not detail on them in this lecture.
- Correspondence problems are also a research topic in our group (bachelor/master thesis topics available).



## Sensor Fusion (6)

### Resampling

- Image fusion often requires to adapt data to common sampling points.
- In this *resampling*, care must be taken again of the band-limitation of functions, possible aliasing effects, etc.
- Resampling may introduce additional blur, giving rise to deconvolution problems.

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# Summary

- The Fourier transform decomposes images into frequences.
- Convolutions are important for describing blurring.
  In the Fourier domain, they become multiplications.
- Aliasing is a sampling artifact that arises from overlapping frequencies due to periodisation in the Fourier domain.
  It can be avoided by obeying the sampling theorem.
- In practise, sampling occurs together with blurring by a PSF.
  It involves separable sampling on a rectangular grid and rotationally symmetric physical blur.
- Sensor fusion is often used to obtain a more comprehensive information from several image sensors. It leads to correspondence problems.



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## References


- S. Webb: *The Physics of Medical Imaging.* Institute of Physics Publishing, Bristol 1988.
  *(CAT, MRI and other methods)*
- A. C. Kak, M. Slaney: *Principles of Computerized Tomographic Imaging.* SIAM, Philadephia 2001.
  *(with sections on the Fourier transform)*
- C. L. Epstein: *Introduction to the Mathematics of Medical Imaging.* Pearson, Upper Saddle River 2003.
  *(with sections on the Fourier transform)*
- R. W. Hamming: *Digital Filters.* Dover, 1998.
  *(classical signal processing book)*